Name:______________________________ Recitation Section:__________

Math 1553 Quiz 4: 1.7, 1.8, 1.9 (10 points, 10 minutes)

Solutions

1. (3 points) In each case, determine whether the set of vectors is linearly dependent or linearly independent. You do not need to justify your answer.

$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\-5\\0\end{pmatrix}, \begin{pmatrix}3\\1\\7\end{pmatrix}\right\}$ linearly dependent **[linearly independent]**

$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}\right\}$ **[linearly dependent]** linearly independent

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-2\\3\end{pmatrix}\right\}$ **[linearly dependent]** linearly independent

(a) Forming a matrix A with the vectors as columns gives 3 pivot columns (no row-reduction required). Alternatively, use the Increasing Span Criterion.
(b) The first and third vectors are multiples of each other.
(c) Three vectors in $\mathbf{R}^2$, so automatically linearly dependent.

2. Suppose A is a 4×3 matrix, with corresponding linear transformation $T(x) = Ax$.

a) Fill in the blank: The domain of T is $\underline{\quad \mathbf{R}^3 \quad}$.

b) True or false (no justification required): It is possible that T is onto.

TRUE **[FALSE]**.

It is impossible for any linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^4$ to be onto. It would require A to have a pivot in every row, which is impossible since a 4×3 matrix has at most 3 pivots.

3. (5 pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation which reflects about the line $y = x$, then rotates counterclockwise by 45°. Find the matrix A so that $T(x) = Ax$ for all x in $\mathbf{R}^2$. Show your work! Write out the numerical values of any trig functions.

Solution.

It's not necessary to use it in this problem, but the matrix for cc rotation 45° is

$$\begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

We examine what happens to e_1 and e_2.

e_1: The reflection sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then rotating cc by 45° gives $T(e_1) = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

e_2: The reflection sends $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then rotating cc by 45° gives $T(e_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Therefore,

$$A = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \boxed{\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}.$$

Alternatively, we could write the reflection matrix $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the rotation matrix $K = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, then take the product KJ:

$$KJ = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$