MATH 1553
PRACTICE MIDTERM 2 (VERSION B)

Name		**Section**	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §1.7 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§1.7 through 2.9.

Problem 1.

[2 points each]

In what follows, A is a matrix, and $T(x) = Ax$ is its matrix transformation.

Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) **T** **F** The zero vector is in the range of T.

b) **T** **F** If A is a non-invertible square matrix, then two of the columns of A are scalar multiples of each other.

c) **T** **F** If A is a 2×5 matrix, then $\text{Nul}A$ is a subspace of $\mathbf{R}^2$.

d) **T** **F** If A has more columns than rows, then T is not onto.

e) **T** **F** If T is one-to-one and onto, then A is invertible

Solution.

a) True: $T(0) = 0$.

b) False: for instance, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is not invertible.

c) False: it is a subspace of $\mathbf{R}^5$.

d) False: however, if A has more rows than columns, then T is not onto.

e) True: the hypothesis implies that A is square, so we can apply the Invertible Matrix Theorem.

Problem 2.

[2 points each]

Which of the following are subspaces of $\mathbf{R}^4$ and **why?**

a) Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 7 \\ 9 \\ 13 \end{pmatrix}, \begin{pmatrix} 144 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

b) Nul $\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 6 & -4 & 2 \\ -9 & 3 & 4 \end{pmatrix}$

c) Col $\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 6 & -4 & 2 \\ -9 & 3 & 4 \end{pmatrix}$

d) $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; xy = zw \right\}$

e) The range of a linear transformation with codomain $\mathbf{R}^4$.

Solution.

a) This is a subspace of $\mathbf{R}^4$: it is a span of three vectors in $\mathbf{R}^4$, and any span is a subspace.

b) This is not a subspace of $\mathbf{R}^4$; it is a subspace of $\mathbf{R}^3$.

c) This is a subspace of $\mathbf{R}^4$: it is the span of three vectors in $\mathbf{R}^4$.

d) This is not a subspace of $\mathbf{R}^4$: it is not closed under addition. For instance,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ are in } V\text{, but } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ is not.}$$

e) This is a subspace of $\mathbf{R}^4$: it is the column space of the associated $4\times ?$ matrix, and any column space is a subspace.

Problem 3.

Consider the matrix A and its reduced row echelon form:

$$\begin{pmatrix} 2 & 4 & 7 & -16 \\ 3 & 6 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

a) [4 points] Find a basis $\{v_1, v_2\}$ for $\operatorname{Col} A$.

b) [3 points] What are $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$?

c) [3 points] Find a basis $\{w_1, w_2\}$ for $\operatorname{Col} A$, such that w_1 is a not scalar multiple of v_1 or v_2, and likewise for w_2. Justify your answer.

Solution.

a) A basis for the column space is given by the pivot columns of A:

$$\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \end{pmatrix} \right\}.$$

b) The rank is the dimension of the column space, which is 2. The rank plus the dimension of the null space equals the number of columns, so the null space has dimension 2 as well.

c) The column space is a 2-dimensional subspace of $\mathbf{R}^2$. Thus $\operatorname{Col} A = \mathbf{R}^2$, so any basis for $\mathbf{R}^2$ works. For example, we can use the standard basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Problem 4.

[5 points each]

Consider the vectors

$$v_1 = \begin{pmatrix}1\\3\\2\end{pmatrix} \qquad v_2 = \begin{pmatrix}-1\\4\\1\end{pmatrix} \qquad v_3 = \begin{pmatrix}1\\h\\5\end{pmatrix}.$$

a) Find the value of h for which $\{v_1, v_2, v_3\}$ is linearly *de*pendent.

b) For this value of h, produce a linear dependence relation among v_1, v_2, v_3.

Solution.

a) Since v_1 and v_2 are not collinear, by the increasing span criterion, $\{v_1, v_2, v_3\}$ is linearly dependent if and only if v_3 is in Span$\{v_1, v_2\}$. This happens if and only if the vector equation

$$x\begin{pmatrix}1\\3\\2\end{pmatrix} + y\begin{pmatrix}-1\\4\\1\end{pmatrix} = \begin{pmatrix}1\\h\\5\end{pmatrix}$$

has a solution. We put it into an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c}1 & -1 & 1\\3 & 4 & h\\2 & 1 & 5\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1 & -1 & 1\\0 & 7 & h-3\\0 & 3 & 3\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1 & -1 & 1\\0 & 3 & 3\\0 & 7 & h-3\end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c}1 & -1 & 1\\0 & 1 & 1\\0 & 7 & h-3\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1 & -1 & 1\\0 & 1 & 1\\0 & 0 & h-10\end{array}\right).$$

This is consistent if and only if $h = 10$, so $\{v_1, v_2, v_3\}$ is linearly dependent if and only if $h = 10$.

b) Our row reduced matrix produces a system of equations:

$$\left(\begin{array}{cc|c}1 & -1 & 1\\0 & 1 & 1\\0 & 0 & 0\end{array}\right) \rightsquigarrow \begin{aligned} x - y &= 1\\ y &= 1.\end{aligned}$$

Hence $x = 2$ and $y = 1$, so

$$2\begin{pmatrix}1\\3\\2\end{pmatrix} + \begin{pmatrix}-1\\4\\1\end{pmatrix} = \begin{pmatrix}1\\10\\5\end{pmatrix}.$$

Strictly speaking, to make this a linear dependence relation, we should put all vectors on the same side of the equation: $2v_1 + v_2 - v_3 = 0$.

Problem 5.

Consider the matrices

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let T and U be the associated linear transformations, respectively

$$T(x) = Ax \qquad U(x) = Bx.$$

a) [2 points] Fill in the boxes:

$$T : \mathbf{R}^{\square} \longrightarrow \mathbf{R}^{\square} \qquad U : \mathbf{R}^{\square} \longrightarrow \mathbf{R}^{\square}.$$

b) [2 points] Is T one-to-one?

c) [3 points] Find the standard matrix for U^{-1}.

d) [3 points] Find the standard matrix for $U \circ T$.

Solution.

a) You can multiply A by vectors in $\mathbf{R}^2$, and the result is a vector in $\mathbf{R}^3$. Therefore,

$$T : \mathbf{R}^2 \longrightarrow \mathbf{R}^3.$$

Likewise, you can multiply B by vectors in $\mathbf{R}^3$, and the result is a vector in $\mathbf{R}^3$. Therefore,

$$U : \mathbf{R}^3 \longrightarrow \mathbf{R}^3.$$

b) The transformation T is one-to-one if and only if A has a pivot in each column, which it does (it is already in row echelon form).

c) The matrix for U^{-1} is B^{-1}. We compute

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -14 \\ 0 & 1 & 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right),$$

so the inverse is

$$B^{-1} = \begin{pmatrix} 1 & 3 & -14 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

d) The matrix for $U \circ T$ is

$$BA = \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}.$$

[Scratch work]