MATH 1553, C. JANKOWSKI
MIDTERM 3

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Write your section number (E6-E9) here: ____________

Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA matches your exam.
- The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work. If you cannot fit your work on the front side of the page, use the back side of the page as indicated.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1.

[Parts a) through f) are worth 2 points each]

a) Suppose A is a 3×3 matrix whose entries are real numbers. How many *distinct* real eigenvalues can A possibly have? Circle all that apply.

(a) 0

(b) 1

(c) 2

(d) 3

The remaining problems are true or false. Answer true if the statement is *always* true. Otherwise, answer false. You do not need to justify your answer. In every case, assume that the entries of the matrix A are real numbers.

b) **T** **F** If A is an $n \times n$ matrix then $\det(-A) = -\det(A)$.

c) **T** **F** If v is an eigenvector of a square matrix A, then $-v$ is also an eigenvector of A.

d) **T** **F** If A is an $n \times n$ matrix and $\lambda = 2$ is an eigenvalue of A, then $\mathrm{Nul}(A - 2I) = \{0\}$.

e) **T** **F** If A is a 3×3 matrix with characteristic polynomial $(3-\lambda)^2(2-\lambda)$, then the eigenvalue $\lambda = 2$ must have geometric multiplicity 1.

f) **T** **F** The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution.

a) Circle (b), (c), and (d). There must be at least one real eigenvalue since n is odd, and there can be two (e.g. $(1-\lambda)^2(5-\lambda)$) or even three.

b) False. Since $\det(cA) = c^n \det(A)$, we see $\det(-A) = (-1)^n \det(A) = \det(A)$ if n is even.

c) True. Straight from the chapter 5 homework. If v is an eigenvector then so is cv for any $c \neq 0$.

d) False. Nul$(A - 2I)$ is the 2-eigenspace, which is never just the zero vector if 2 is an eigenvalue.

e) True. The geometric multiplicity of an eigenvalue is always at least 1 but never more than the algebraic multiplicity (which here is 1).

f) True. Every 2×2 matrix with eigenvalues $\lambda = 2$ and $\lambda = 3$ is similar to $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and to $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ depending upon how you place the eigenvectors, so the matrices are similar. Or you could observe $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = P \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}$ where $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Extra space for scratch work on problem 1

Problem 2. [10 points]

Short answer. For (a) and (b), show any brief computations. For (c), (d), and (e), you do not need to justify your answer. In each case, assume the entries of A and B are real numbers.

a) Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 7 \end{pmatrix}$, and define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$. Find the area of $T(S)$, if S is a triangle in $\mathbf{R}^2$ with area 2.

b) Find $\det \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 5 & 0 \end{pmatrix}$.

c) Write a 2×2 matrix A which is not diagonalizable and not invertible.

d) Give an example of 2×2 matrices A and B which have the same characteristic polynomial but are not similar.

e) Write a diagonalizable 3×3 matrix A whose only eigenvalue is $\lambda = 2$.

Solution.

a) $|\det(A)|\mathrm{Vol}(S) = |-7-1| \cdot 2 = 16$.

b) The matrix is triangular with one row swap, so its determinant is

$$-(1 \cdot 3 \cdot -1 \cdot 5 \cdot 2) = 30.$$

c) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

d) For example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

e) There is only one such matrix: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Extra space for work on problem 2

Problem 3. [10 points]

Let $A = \begin{pmatrix} 2 & -4 \\ 1 & 2 \end{pmatrix}$.

a) Find the eigenvalues of A.

b) Let λ be the eigenvalue of A whose imaginary part is negative. Find an eigenvector of A corresponding to λ.

c) Find a matrix C which is similar to A and represents a composition of scaling and rotation.

d) What is the scaling factor for C?

e) Find the angle of rotation for C.
(do not leave your answer in terms of arctan; the answer is a standard angle).

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} 2-\lambda & -4 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2+4 = \lambda^2-4\lambda+8, \qquad \lambda = \frac{4 \pm \sqrt{4^2-32}}{2} = \frac{4 \pm \sqrt{-16}}{2} = 2\pm 2i.$$

b) $\left(A-(2-2i)I \quad 0\right) = \begin{pmatrix} 2i & -4 & 0 \\ 1 & 2i & 0 \end{pmatrix} = \left(\begin{array}{cc|c} 2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right)$. One eigenvector is $v = \begin{pmatrix} -4 \\ -2i \end{pmatrix}$, or $\begin{pmatrix} 4 \\ 2i \end{pmatrix}$. Alternatively, we could row reduce $\left(\begin{array}{cc|c} 2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1=R_1/(2i)} \begin{pmatrix} 1 & 2i & 0 \end{pmatrix}$, so $\begin{pmatrix} -2i \\ 1 \end{pmatrix}$ is an eigenvector. Really, any nonzero multiple of $\begin{pmatrix} -4 \\ -2i \end{pmatrix}$ is an eigenvector.

c) We have C_1 and C_2 as possibilities for C, done as follows.
If we use $\lambda = 2-2i$ then $C_1 = \begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$.
However, if we use $\lambda = 2+2i$ then $C_2 = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$.

d) $|\lambda| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$ (it is fine if the student leaves it as $\sqrt{8}$).

e) Use the $-\arg(\lambda)$ formula or just use knowledge of rotations (which we do below).

$$C_1 = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} = 2\sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \cos(\theta_1) = \sin(\theta_1) = \frac{1}{\sqrt{2}}, \text{ so } \theta_1 = \frac{\pi}{4}.$$

If the student used C_2, then the angle is different:

$$C_2 = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} = 2\sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad \cos(\theta_2) = \frac{1}{\sqrt{2}}, \ \sin(\theta_2) = -\frac{1}{\sqrt{2}}, \text{ so } \theta_2 = -\frac{\pi}{4}.$$

Extra space for work on problem 3

Problem 4.

[9 points]

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

a) Find the eigenvalues of A, and find a basis for each eigenspace.

b) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix P so that $A = PDP^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1)^6 \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (-1-\lambda)((2-\lambda)^2 - 9)$$

$$= (-1-\lambda)(\lambda^2 - 4\lambda - 5) = -(\lambda+1)^2(\lambda - 5).$$

So $\lambda = -1$ and $\lambda = 5$ are the eigenvalues.

$\underline{\lambda = -1}$: $$\left(A + I \mid 0\right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 = R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

with solution $x_1 = -x_2$, $x_2 = x_2$, $x_3 = 0$. The (-1)-eigenspace has basis $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

$\underline{\lambda = 5}$:

$$\left(A - 5I \mid 0\right) = \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \xrightarrow[R_3 = R_3/(-6)]{R_2 = R_2 + R_1} \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{then } R_2 \leftrightarrow R_3,\ R_1/(-3)]{R_1 = R_1 - R_3,\ R_2 = R_2 - 5R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

with solution $x_1 = x_2,\ x_2 = x_2,\ x_3 = 0$. The 5-eigenspace has basis $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

b) A is a 3×3 matrix that only admits 2 linearly independent eigenvectors, so A is not diagonalizable.

Extra space for work on problem 4

Problem 5.

[9 points]

Parts (a) and (b) are not related.

a) Find a 2×2 matrix A whose 2-eigenspace is the line $y = 2x$ and whose (-1)-eigenspace is the line $y = 3x$. Be sure your work is clear.

b) Let B be a 4×4 matrix satisfying $\det(B) = 2$, and let

$$C = \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ -1 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Find $\det(CB^{-1})$.

Solution.

a) We want $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to be an eigenvector for eigenvalue 2, and we want $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ to be an eigenvector for eigenvalue -1. This means $A = PDP^{-1}$, where $P = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -3 \\ 18 & -7 \end{pmatrix}.$$

b) We use the cofactor expansion along the second column:

$$\det(C) = 1(-1)^{3+2}\det\begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} = -1 \cdot 2 \cdot (-1)^2 \det\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} = -2 \cdot (-2-3) = 10.$$

Therefore,

$$\det(CB^{-1}) = \det(C)\det(B^{-1}) = 10 \cdot \frac{1}{2} = 5.$$

Extra space for work on problem 5