Math 1553 Worksheet §2.8 (and some 2.9)

1. Find bases for the column space and the null space of

$$A = \begin{pmatrix} 0 & 1 & -3 & 1 & 0 \\ 1 & -1 & 8 & -7 & 1 \\ -1 & -2 & 1 & 4 & -1 \end{pmatrix}.$$

Solution.

Finding a basis for Nul$A$ means finding the parametric vector form of the solution to $Ax = 0$. First we row reduce:

$$\begin{pmatrix} 0 & 1 & -3 & 1 & 0 \\ 1 & -1 & 8 & -7 & 1 \\ -1 & -2 & 1 & 4 & -1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 5 & -6 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so x_3, x_4, x_5 are free, and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -5x_3 + 6x_4 - x_5 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for Nul A is $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

To find a basis for Col A, we use the pivot columns as they were written in the *original* matrix A, *not its RREF*. These are the first two columns:

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}.$$

2. Consider the following vectors in $\mathbf{R}^3$:

$$b_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \qquad u = \begin{pmatrix} 1 \\ 10 \\ 7 \end{pmatrix}$$

Let $V = \text{Span}\{b_1, b_2\}$.

a) Explain why $\mathcal{B} = \{b_1, b_2\}$ is a basis for V.

b) Determine if u is in V.

c) Find a vector b_3 such that $\{b_1, b_2, b_3\}$ is a basis of $\mathbf{R}^3$.

Solution.

a) A quick check shows that b_1 and b_2 are linearly independent (verify!), and we already know they span V, so $\{b_1, b_2\}$ is a basis for V.

b) u is in V if and only if $c_1 b_1 + c_2 b_2 = u$ for some c_1 and c_2 (in which case $[u]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ looking ahead to problem 5(b)). We form the augmented matrix $\left(\begin{array}{cc|c} b_1 & b_2 & u \end{array}\right)$ and see if the system is consistent.

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 2 & 4 & 10 \\ 2 & 3 & 7 \end{array}\right) \xrightarrow[R_3=R_3-R_1]{R_2=R_2-R_1} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 3 & 9 \\ 0 & 2 & 6 \end{array}\right) \xrightarrow[R_2=R_2/3]{R_3=R_3-\frac{2}{3}R_2} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right).$$

The right column is not a pivot column, so the system is consistent, therefore u is in $\mathrm{Span}\{b_1, b_2\}$: in fact, $u = -b_1 + 3b_2$.

c) If we choose b_3 which is not in $\mathrm{Span}\{b_1, b_2\}$, then $\{b_1, b_2, b_3\}$ is linearly independent by the increasing span criterion. Any three linearly independent vectors span $\mathbf{R}^3$: the matrix with columns b_1, b_2, b_3 is square, so if there is a pivot in every column, then there is a pivot in every row.

We could choose $b_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, since $\left(\begin{array}{cc|c} b_1 & b_2 & b_3 \end{array}\right)$ is inconsistent:

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 2 & 4 & 0 \\ 2 & 3 & 0 \end{array}\right) \xrightarrow[R_3=R_3-R_1]{R_2=R_2-R_1} \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 3 & -1 \\ 0 & 2 & -1 \end{array}\right) \xrightarrow{R_3 = R_3 - \frac{2}{3}R_2} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -1/3 \end{array}\right).$$

3. For (a) and (b), answer "yes" if the statement is always true, "no" if it is always false, and "maybe" otherwise.

a) If A is an $n \times n$ matrix and $\mathrm{Col}\ A = \mathbf{R}^n$, then $Ax = 0$ has a nontrivial solution.

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

c) Give an example of 2×2 matrix whose column space is the same as its null space.

Solution.

a) No. Since $\mathrm{Col}(A) = \mathbf{R}^n$, the linear transformation $T(x) = Ax$ from $\mathbf{R}^n$ to $\mathbf{R}^n$ is onto, hence T is one-to-one, so $Ax = 0$ has only the trivial solution.

b) Maybe. If $Ax = 0$ has only the trivial solution and $m = n$, then A is invertible, so the columns of A are linearly independent and span $\mathbf{R}^m$.

If $m > n$ then the statement is false. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its columns form only a 2-plane within $\mathbf{R}^3$.

c) Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Its null space and column space are $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

4. In each case, determine whether the given set is a subspace of $\mathbf{R}^4$. If it is a subspace, justify why. If it is not a subspace, state a subspace property that it fails.

a) $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x + y = 0 \text{ and } z + w = 0 \right\}$

b) $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; xy - zw = 0 \right\}$

Solution.

a) The condition "$x + y = 0$ and $z + w = 0$" means that the vectors in V are the solutions to the system of homogeneous equations

$$\begin{aligned} x + y \qquad\quad &= 0 \\ z + w &= 0. \end{aligned}$$

In other words, V is the null space of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

A null space is automatically a subspace, so V is a subspace.

Alternatively, we can verify the subspace properties:

(1) The zero vector is in V, since $0+0=0$ and $0+0=0$.

(2) If $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$ are in V. Compute $u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}$.
Are $(x_1 + x_2) + (y_1 + y_2) = 0$ and $(z_1 + z_2) + (w_1 + w_2) = 0$? Yes:

$$\begin{aligned} (x_1 + x_2) + (y_1 + y_2) &= (x_1 + y_1) + (x_2 + y_2) = 0 + 0 = 0, \\ (z_1 + z_2) + (w_1 + w_2) &= (z_1 + w_1) + (z_2 + w_2) = 0 + 0 = 0. \end{aligned}$$

(3) If $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ is in V then so is cu for any scalar:

$$cx_1 + cy_1 = c(x_1 + y_1) = c(0) = 0, \qquad cz_1 + cw_1 = c(z_1 + w_1) = c(0) = 0.$$

b) Not a subspace. Note $u = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $v = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are in W, but $u+v$ is not in W.

$$u+v = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \qquad 1\cdot 1 - 0\cdot 0 = 1 \neq 0. \quad (W \text{ is not closed under addition})$$

5. This problem covers section 2.9. Parts (a), (b), and (c) are unrelated to each other.

a) True or false: If A is a 3×100 matrix of rank 2, then $\dim(\mathrm{Nul}A) = 97$.

b) For u and $\mathcal{B}$ from problem 2, find $[u]_{\mathcal{B}}$ (the $\mathcal{B}$-coordinates of u).

c) Let $\mathcal{D} = \left\{\begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\}$, and suppose $[x]_{\mathcal{D}} = \begin{pmatrix}-1\\3\end{pmatrix}$. Find x.

Solution.

a) No. By the Rank Theorem, $\mathrm{rank}(A) + \dim(\mathrm{Nul}A) = 100$, so $\dim(\mathrm{Nul}A) = 98$.

b) u is in V if and only if $c_1 b_1 + c_2 b_2 = u$ for some c_1 and c_2, in which case $[u]_{\mathcal{B}} = \begin{pmatrix}c_1\\c_2\end{pmatrix}$. We form the augmented matrix $\left(\begin{array}{cc|c} b_1 & b_2 & u\end{array}\right)$ and solve:

$$\left(\begin{array}{cc|c} 2 & 1 & 1\\ 2 & 4 & 10\\ 2 & 3 & 7\end{array}\right) \xrightarrow[R_3=R_3-R_1]{R_2=R_2-R_1} \left(\begin{array}{cc|c} 2 & 1 & 1\\ 0 & 3 & 9\\ 0 & 2 & 6\end{array}\right) \xrightarrow[R_2=R_2/3]{R_3=R_3-\frac{2}{3}R_2} \left(\begin{array}{cc|c} 2 & 1 & 1\\ 0 & 1 & 3\\ 0 & 0 & 0\end{array}\right) \xrightarrow[\text{then } R_1=R_1/2]{R_1=R_1-R_2} \left(\begin{array}{cc|c} 1 & 0 & -1\\ 0 & 1 & 3\\ 0 & 0 & 0\end{array}\right).$$

We found $c_1 = -1$ and $c_2 = 3$. This means $-b_1 + 3b_2 = u$, so $[u]_{\mathcal{B}} = \begin{pmatrix}-1\\3\end{pmatrix}$.

c) From $[x]_{\mathcal{D}} = \begin{pmatrix}-1\\3\end{pmatrix}$, we have

$$x = -d_1 + 3d_2 = -\begin{pmatrix}-2\\1\end{pmatrix} + 3\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}9\\3\end{pmatrix} = \begin{pmatrix}11\\2\end{pmatrix}.$$