Math 1553 Worksheet §1.7, 1.8, 1.9

Solutions

1. Justify why each of the following true statements can be checked without row reduction.

a) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$ is linearly independent.

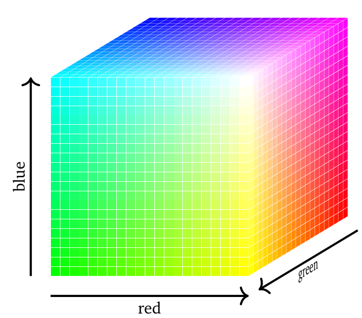
b) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is linearly dependent.

Solution.

a) Here's how to eyeball linear independence. Since the first coordinate of $\begin{pmatrix}3\\3\\4\end{pmatrix}$ is nonzero, $\begin{pmatrix}3\\3\\4\end{pmatrix}$ cannot be in the span of $\left\{\begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$. And $\begin{pmatrix}0\\10\\20\end{pmatrix}$ is not in the span of $\left\{\begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$ because it is not a multiple. Hence the span gets bigger each time you add a vector, so they're linearly independent.

b) Any four vectors in $\mathbf{R}^3$ are linearly dependent; you don't need row reduction for that.

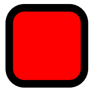
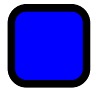
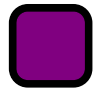
2. Every color on my computer monitor is a vector in $\mathbf{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors $v_1, v_2, \ldots, v_p$, we can form a "weighted average" of these colors by making a linear combination

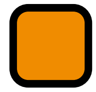
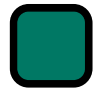
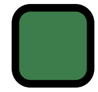
$$v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:

$\frac{1}{2}$  $+\ \frac{1}{2}$  $=$ 

Consider the colors on the right. Are these colors linearly independent? What does this tell you about the colors?

$$\begin{pmatrix}240\\140\\0\end{pmatrix} \quad \begin{pmatrix}0\\120\\100\end{pmatrix} \quad \begin{pmatrix}60\\125\\75\end{pmatrix}$$

Solution.

The vectors

$$\begin{pmatrix}240\\140\\0\end{pmatrix}, \quad \begin{pmatrix}0\\120\\100\end{pmatrix}, \quad \begin{pmatrix}60\\125\\75\end{pmatrix}$$

are linearly independent if and only if the vector equation

$$x\begin{pmatrix}240\\140\\0\end{pmatrix} + y\begin{pmatrix}0\\120\\100\end{pmatrix} + z\begin{pmatrix}60\\125\\75\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

has only the trivial solution. This translates into the matrix (we don't need to augment since it's a homogeneous system)

$$\begin{pmatrix}240 & 0 & 60\\140 & 120 & 125\\0 & 100 & 75\end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix}1 & 0 & .25\\0 & 1 & .75\\0 & 0 & 0\end{pmatrix} \xrightarrow{\text{parametric}} \begin{aligned} x &= -.25z\\ y &= -.75z\end{aligned}$$

Hence the vectors are linearly *de*pendent; taking $z = 1$ gives the linear dependence relation

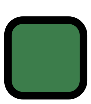
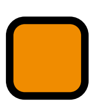
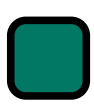
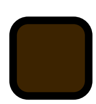
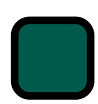
$$-\frac{1}{4}\begin{pmatrix}240\\140\\0\end{pmatrix} - \frac{3}{4}\begin{pmatrix}0\\120\\100\end{pmatrix} + \begin{pmatrix}60\\125\\75\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Rearranging gives

$$\begin{pmatrix}60\\125\\75\end{pmatrix} = \frac{1}{4}\begin{pmatrix}240\\140\\0\end{pmatrix} + \frac{3}{4}\begin{pmatrix}0\\120\\100\end{pmatrix}.$$

In terms of colors:

$$\begin{pmatrix}60\\125\\75\end{pmatrix} = \frac{1}{4}\begin{pmatrix}240\\140\\0\end{pmatrix} + \frac{3}{4}\begin{pmatrix}0\\120\\100\end{pmatrix} = \begin{pmatrix}60\\35\\0\end{pmatrix} + \begin{pmatrix}0\\90\\75\end{pmatrix}$$

 $=\ \frac{1}{4}$  $+\ \frac{3}{4}$  $=$  $+$ 

3. Let A be a 3×4 matrix with column vectors v_1, v_2, v_3, v_4. Suppose that $v_2 = 2v_1 - 3v_4$. Find one non-trivial solution to the equation $Ax = 0$.

Solution.

From the linear dependence condition we were given, we get

$$-2v_1 + v_2 + 3v_4 = 0.$$

This vector equation is just

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{so} \quad A\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad \text{Thus, } x = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} \text{ is one solution.}$$

4. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) Counterclockwise rotation by $32°$ in $\mathbf{R}^2$.

b) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (z, x)$.

c) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (0, x)$.

d) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix}$.

e) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution.

a) This is both one-to-one and onto. If v is any vector in $\mathbf{R}^2$, then there is one and only one vector w such that $T(w) = v$, namely, the vector that is rotated $-32°$ from v.

b) This is onto. If (a, b) is any vector in the codomain $\mathbf{R}^2$, then $(a, b) = T(b, 0, a)$, so (a, b) is in the range. It is not one-to-one though: indeed, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$.

c) This is not onto. There is no (x, y, z) such that $T(x, y, z) = (1, 0)$. It is not one-to-one: for instance, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$.

d) The transformation T with matrix A is onto if and only if A has a pivot in every *row*, and it is one-to-one if and only if A has a pivot in every *column*. So we row reduce:

$$A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This has a pivot in every column, so T is one-to-one. It does not have a pivot in every row, so it is not onto. To find a specific vector b in $\mathbf{R}^3$ which is not in the image of T, we have to find a $b = (b_1, b_2, b_3)$ such that the matrix equation $Ax = b$ is inconsistent. We row reduce again:

$$\left(A \mid b\right) = \left(\begin{array}{cc|c} 1 & 6 & b_1 \\ -1 & 2 & b_2 \\ 2 & -1 & b_3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & \text{don't care} \\ 0 & 1 & \text{don't care} \\ 0 & 0 & -3b_1 + 13b_2 + 8b_3 \end{array}\right).$$

Hence any b_1, b_2, b_3 for which $-3b_1 + 13b_2 + 8b_3 \neq 0$ will make the equation $Ax = b$ inconsistent. For instance, $b = (1, 0, 0)$ is not in the range of T.

e) This matrix is already row reduced. We can see that does not have a pivot in every row *or* in every column, so it is neither onto nor one-to-one. In fact, if $T(x) = Ax$ then

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_3 \\ 0 \end{pmatrix},$$

so we can see that $(0, 0, 1)$ is not in the range of T, and that

$$T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

5. For each matrix A, describe what the associated matrix transformation T does to $\mathbf{R}^3$ geometrically.

$$\textbf{a)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution.

a) We compute

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ z \end{pmatrix}.$$

This is the reflection over the plane $y = x$.

b) We compute

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}.$$

This is projection onto the z-axis.

6. The second little pig has decided to build his house out of sticks. The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis, and then projected onto the xy-plane. Find the matrix for this transformation.

Solution.

To compute the matrix for T, we have to compute $T(e_1), T(e_2)$, and $T(e_3)$. To see the picture, let's put ourselves above the xy-plane, looking downward.

$$T(e_1) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad T(e_2) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}.$$

Rotating e_3 around the z-axis does nothing, and projecting onto the xy-plane sends it to zero, so $T(e_3) = 0$. Therefore, the matrix for T is

$$\begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 & 0\\1 & 1 & 0\\0 & 0 & 0\end{pmatrix}.$$