Section 5.3

Diagonalization

Many real-word linear algebra problems have the form:

$$v_1 = Av_0, \quad v_2 = Av_1 = A^2 v_0, \quad v_3 = Av_2 = A^3 v_0, \quad \ldots \quad v_n = Av_{n-1} = A^n v_0.$$

This is called a **difference equation**.

Our toy example about rabbit populations had this form.

The question is, what happens to v_n as $n \to \infty$?

- Taking powers of diagonal matrices is easy!
- Taking powers of *diagonalizable* matrices is still easy!
- Diagonalizing a matrix is an eigenvalue problem.

Powers of Diagonal Matrices

If D is diagonal, then D^n is also diagonal; its diagonal entries are the nth powers of the diagonal entries of D:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}, \quad D^3 = \begin{pmatrix} 8 & 0 \\ 0 & 27 \end{pmatrix}, \quad \ldots \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}.$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad D^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix},$$

$$\ldots \quad D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{pmatrix}$$

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

Example

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. Compute A^n.

In §5.2 lecture we saw that A is similar to a diagonal matrix:

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PD^2P^{-1} \\ A^3 &= (PDP^{-1})(PD^2P^{-1}) = PD(P^{-1}P)D^2P^{-1} = PDID^2P^{-1} = PD^3P^{-1} \\ &\vdots \\ A^n &= PD^nP^{-1} \end{aligned}$$

Closed formula in terms of n: easy to compute

Therefore

$$A^n = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2^{n+1} - 3^n & -2^{n+1} + 2 \cdot 3^n \\ 2^n - 3^n & -2^n + 2 \cdot 3^n \end{pmatrix}.$$

Diagonalizable Matrices

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = PDP^{-1} \quad \text{for } D \text{ diagonal.}$$

Important

If $A = PDP^{-1}$ for $D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$ then

$$A^k = PD^KP^{-1} = P \begin{pmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{pmatrix} P^{-1}.$$

So diagonalizable matrices are easy to raise to any power.

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In this case, $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Corollary

← a theorem that follows easily from another theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

The Corollary is true because eigenvectors with distinct eigenvalues are always linearly independent. We will see later that a diagonalizable matrix need not have n distinct eigenvalues though.

Problem: Diagonalize $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Therefore the eigenvalues are 2 and 3. Let's compute some eigenvectors:

$$(A - 2I)x = 0 \iff \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = 2y$, so $v_1 = \binom{2}{1}$ is an eigenvector with eigenvalue 2.

$$(A - 3I)x = 0 \iff \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = y$, so $v_2 = \binom{1}{1}$ is an eigenvector with eigenvalue 3.

The eigenvectors v_1, v_2 are linearly independent, so the Diagonalization Theorem says

$$A = PDP^{-1} \quad \text{for} \quad P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2).$$

Therefore the eigenvalues are 1 and 2, with respective multiplicities 2 and 1.
Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix} x = 0 \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric vector form is

$$\begin{aligned} x &= y \\ y &= y \\ z &= \quad\quad z \end{aligned} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence a basis for the 1-eigenspace is

$$\mathcal{B}_1 = \{v_1, v_2\} \quad \text{where} \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Now let's compute the 2-eigenspace:

$$(A - 2I)x = 0 \iff \begin{pmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = 3z, y = 2z$, so an eigenvector with eigenvalue 2 is

$$v_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The eigenvectors v_1, v_2, v_3 are linearly independent: v_1, v_2 form a basis for the 1-eigenspace, and v_3 is not contained in the 1-eigenspace. Therefore the Diagonalization Theorem says

$$A = PDP^{-1} \quad \text{for} \quad P = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note: In this case, there are three linearly independent eigenvectors, but only two distinct eigenvalues.

Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

This is an upper-triangular matrix, so the only eigenvalue is 1. Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = 0.$$

This is row reduced, but has only one free variable x; a basis for the 1-eigenspace is $\{\binom{1}{0}\}$. So *all eigenvectors* of A are multiples of $\binom{1}{0}$.

Conclusion: A has only one linearly independent eigenvector, so by the "only if" part of the diagonalization theorem, A is not diagonalizable.

Poll

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Matrix A is not diagonalizable: its only eigenvalue is 2, and its 2-eigenspace is spanned by $\binom{1}{0}$.

Matrix B is diagonalizable because it is a 2×2 matrix with distinct eigenvalues.

Matrix C is already diagonal!

How to diagonalize a matrix A:

1. Find the eigenvalues of A using the characteristic polynomial.
2. For each eigenvalue λ of A, compute a basis $\mathcal{B}_\lambda$ for the λ-eigenspace.
3. If there are fewer than n total vectors in the union of all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the n vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for v_i.

Why is the Diagonalization Theorem true?

A diagonalizable implies A has n linearly independent eigenvectors: Suppose $A = PDP^{-1}$, where D is diagonal with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v_1, v_2, \ldots, v_n$ be the columns of P. They are linearly independent because P is invertible. So $Pe_i = v_i$, hence $P^{-1}v_i = e_i$.

$$Av_i = PDP^{-1}v_i = PDe_i = P(\lambda_i e_i) = \lambda_i Pe_i = \lambda_i v_i.$$

Hence v_i is an eigenvector of A with eigenvalue λ_i. So the columns of P form n linearly independent eigenvectors of A, and the diagonal entries of D are the eigenvalues.

A has n linearly independent eigenvectors implies A is diagonalizable: Suppose A has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let P be the invertible matrix with columns $v_1, v_2, \ldots, v_n$. Let $D = P^{-1}AP$.

$$De_i = P^{-1}APe_i = P^{-1}Av_i = P^{-1}(\lambda_i v_i) = \lambda_i P^{-1}v_i = \lambda_i e_i.$$

Hence D is diagonal, with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Solving $D = P^{-1}AP$ for A gives $A = PDP^{-1}$.

Non-Distinct Eigenvalues

Definition

Let λ be an eigenvalue of a square matrix A. The **geometric multiplicity** of λ is the dimension of the λ-eigenspace.

Theorem

Let λ be an eigenvalue of a square matrix A. Then

$$1 \leq (\text{the geometric multiplicity of } \lambda) \leq (\text{the algebraic multiplicity of } \lambda).$$

The proof is beyond the scope of this course.

Corollary

Let λ be an eigenvalue of a square matrix A. If the algebraic multiplicity of λ is 1, then the geometric multiplicity is also 1.

The Diagonalization Theorem (Alternate Form)

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of A equals n.
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n, and *the geometric multiplicity equals the algebraic multiplicity* of each eigenvalue.

Example

If A has n distinct eigenvalues, then the algebraic multiplicity of each equals 1, hence so does the geometric multiplicity, and therefore A is diagonalizable.

For example, $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ has eigenvalues 2 and 3, so it is diagonalizable.

Example

The matrix $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -(\lambda - 1)^2(\lambda - 2).$$

The algebraic multiplicities of 1 and 2 are 2 and 1, respectively. They sum to 3.

We showed before that the geometric multiplicity of 1 is 2 (the 1-eigenspace has dimension 2). The eigenvalue 2 automatically has geometric multiplicity 1.

Hence the geometric multiplicities add up to 3, so A is diagonalizable.

Non-Distinct Eigenvalues

Another example

Example

The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $f(\lambda) = (\lambda - 1)^2$.

It has one eigenvalue 1 of algebraic multiplicity 2.

We showed before that the geometric multiplicity of 1 is 1 (the 1-eigenspace has dimension 1).

Since the geometric multiplicity is smaller than the algebraic multiplicity, the matrix is *not* diagonalizable.

Applications to Difference Equations

Let $D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$.

Fix a vector v_0, and let $v_1 = Dv_0$, $v_2 = Dv_1$, etc., so $v_n = D^n v_0$.

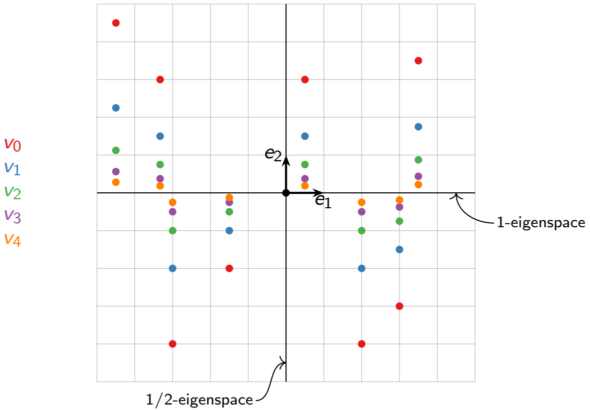
Question: What happens to the v_i's for different choices of v_0?

Answer: Note that D is diagonal, so

$$D^n \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1^n & 0 \\ 0 & 1/2^n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b/2^n \end{pmatrix}.$$

So the x-coordinate of v_n equals the x-coordinate of v_0, and the y-coordinate gets halved every time.

$$D\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b/2 \end{pmatrix}$$



So all vectors get "sucked into the x-axis," which is the 1-eigenspace.

Let $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$.

Fix a vector v_0, and let $v_1 = Av_0$, $v_2 = Av_1$, etc., so $v_n = A^n v_0$.

Question: What happens to the v_i's for different choices of v_0?

Answer: We want to compute powers of A, so this is a diagonalization question. The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\Big(\lambda - \frac{1}{2}\Big).$$

We compute eigenvectors with eigenvalues 1 and $1/2$ to be, respectively,

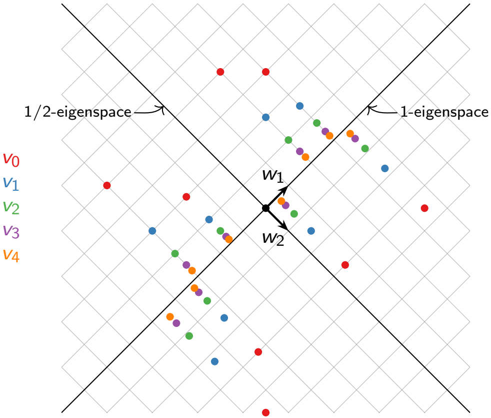
$$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, $\quad A = PDP^{-1} \quad$ for $\quad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$.

This is the same matrix D from before. Hence

$$v_n = A^n v_0 = PD^n P^{-1} v_0.$$

Recall: $A^n = PD^nP^{-1}$ acts on the usual coordinates of v_0 in the same way that D^n acts on the $\mathcal{B}$-coordinates, where $\mathcal{B} = \{w_1, w_2\}$.



So all vectors get "sucked into the 1-eigenspace."