Section 2.8

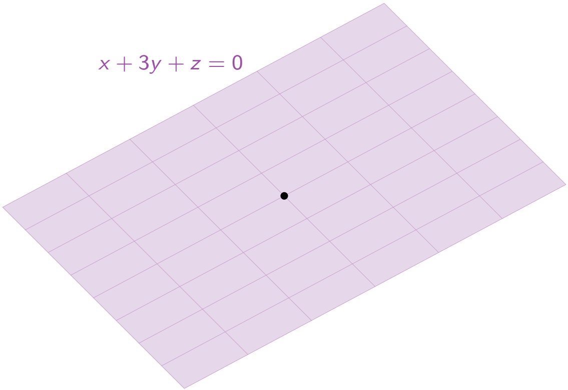
Subspaces of $\mathbf{R}^n$

Motivation

Today we will discuss **subspaces** of $\mathbf{R}^n$.

A subspace turns out to be the same as a span, except we don't know *which* vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbf{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.



Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Fast-forward

Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven't computed what those vectors are yet.

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What does this mean?

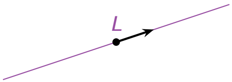
- If v is in V, then all scalar multiples of v are in V by (3). That is, the line through v is in V.
- If u, v are in V, then xu and yv are in V for scalars x, y by (3). So $xu + yv$ is in V by (2). So $\text{Span}\{u, v\}$ is contained in V.
- Likewise, if $v_1, v_2, \ldots, v_n$ are all in V, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ is contained in V.

> A subspace V contains the span of any set of vectors in V.

Examples

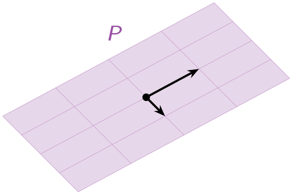
Example

A line L through the origin: this contains the span of any vector in L.



Example

A plane P through the origin: this contains the span of any vectors in P.



Example

All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

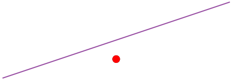
Example

The subset $\{0\}$: this subspace contains only one vector.

Note these are all pictures of spans! (Line through origin, plane through origin, space through origin, etc.)

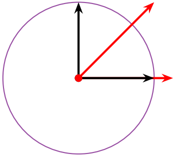
Non-Example

A line L (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.



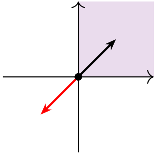
Non-Example

A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."



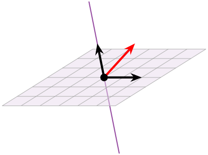
Non-Example

The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.



Non-Example

A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails: 2 only.



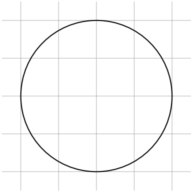
Subsets and Subspaces

They aren't the same thing

A **subset** of $\mathbf{R}^n$ is any collection of vectors whatsoever.

All of the non-examples are still subsets.

A **subspace** is a special kind of subset, which satisfies the three defining properties.



Subset: *yes*
Subspace: *no*

Spans are Subspaces

Theorem

Any Span$\{v_1, v_2, \ldots, v_n\}$ is a subspace.

> **!!!**
>
> Every subspace is a span, and every span is a subspace.

Definition

If $V = \text{Span}\{v_1, v_2, \ldots, v_n\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_n$.

Check:

1. $0 = 0v_1 + 0v_2 + \cdots + 0v_n$ is in the span.
2. If, say, $u = 3v_1 + 4v_2$ and $v = -v_1 - 2v_2$, then

   $$u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2$$

   is also in the span.
3. Similarly, if u is in the span, then so is cu for any scalar c.

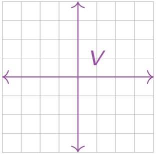
Subspaces

Verification

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; ab = 0 \right\}$. Let's check if V is a subspace or not.

1. Does V contain the zero vector? $\binom{a}{b} = \binom{0}{0} \implies ab = 0$ ✔

3. Is V closed under scalar multiplication?
   - Let $\binom{a}{b}$ be in V.
   - *This means:* a and b are numbers such that $ab = 0$.
   - Let c be a scalar. Is $c\binom{a}{b} = \binom{ca}{cb}$ in V?
   - *This means:* $(ca)(cb) = 0$.
   - Well, $(ca)(cb) = c^2(ab) = c^2(0) = 0$ ✔



2. Is V closed under addition?
   - Let $\binom{a}{b}$ and $\binom{a'}{b'}$ be in V.
   - *This means:* $ab = 0$ and $a'b' = 0$.
   - Is $\binom{a}{b} + \binom{a'}{b'} = \binom{a+a'}{b+b'}$ in V?
   - *This means:* $(a + a')(b + b') = 0$.
   - This is not true for all such a, a', b, b': for instance, $\binom{1}{0}$ and $\binom{0}{1}$ are in V, but their sum $\binom{1}{0} + \binom{0}{1} = \binom{1}{1}$ is not in V, because $1 \cdot 1 \neq 0$. ✗

We conclude that V is *not* a subspace. A picture is above. (It doesn't look like a span.)

Column Space and Null Space

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

- The **column space** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is written $\operatorname{Col} A$.
- The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:
$$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$
  This is a subspace of $\mathbf{R}^n$.

The column space is defined as a span, so we know it is a subspace. It is the range (as opposed to the codomain) of the transformation $T(x) = Ax$.

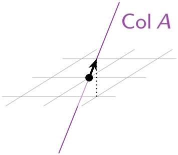
Check that the null space is a subspace:

1. 0 is in $\operatorname{Nul} A$ because $A0 = 0$.
2. If u and v are in $\operatorname{Nul} A$, then $Au = 0$ and $Av = 0$. Hence
$$A(u + v) = Au + Av = 0,$$
   so $u + v$ is in $\operatorname{Nul} A$.
3. If u is in $\operatorname{Nul} A$, then $Au = 0$. For any scalar c, $A(cu) = cAu = 0$. So cu is in $\operatorname{Nul} A$.

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the column space:

$$\text{Col}\,A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$



This is a line in $\mathbf{R}^3$.

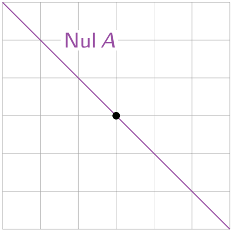
Let's compute the null space:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix}.$$

This zero if and only if $x = -y$. So

$$\text{Nul}\,A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \mid y = -x \right\}.$$

This defines a line in $\mathbf{R}^2$:



The Null Space is a Span

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question

How to find vectors which span the null space?

Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, say, x_3 and x_4 are the free variables. So

$$\text{Nul}\, A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §1.5 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

The Null Space is a Span

Example, revisited

Find vector(s) that span the null space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

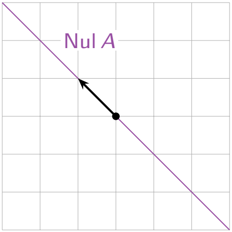
The reduced row echelon form is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This gives the equation $x + y = 0$, or

$$\begin{aligned} x &= -y \\ y &= y \end{aligned} \quad \xrightarrow{\text{parametric vector form}} \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The null space is

$$\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$



How do you check if a subset is a subspace?

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?

Basis of a Subspace

What is the *smallest number* of vectors that are needed to span a subspace?

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Note the big red border here

Why is a basis the smallest number of vectors needed to span?

Recall: *linearly independent* means that every time you add another vector, the span gets bigger.

Hence, if we remove any vector, the span gets *smaller*: so any smaller set can't span V.

Important

A subspace has *many different* bases, but they all have the same number of vectors (see the exercises in §2.9).

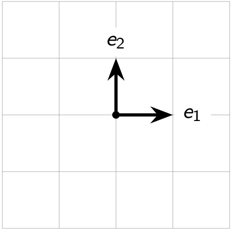
Bases of $\mathbf{R}^2$

Question

What is a basis for $\mathbf{R}^2$?

We need two vectors that *span* $\mathbf{R}^2$ and are *linearly independent*. $\{e_1, e_2\}$ is one basis.

1. They span: $\binom{a}{b} = ae_1 + be_2$.
2. They are linearly independent because they are not collinear.

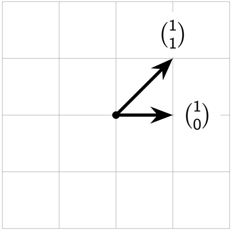


Question

What is another basis for $\mathbf{R}^2$?

Any two nonzero vectors that are not collinear. $\left\{\binom{1}{0}, \binom{1}{1}\right\}$ is also a basis.

1. They span: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every row.
2. They are linearly independent: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every column.



Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix}1\\0\\ \vdots\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\ \vdots\\0\\0\end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix}0\\0\\ \vdots\\1\\0\end{pmatrix}, \quad e_n = \begin{pmatrix}0\\0\\ \vdots\\0\\1\end{pmatrix}$$

are a basis for $\mathbf{R}^n$. The identity matrix has columns $e_1, e_2, \ldots, e_n$.

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

has a pivot in every row and every column, i.e. if A is *invertible*.

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that $\mathcal{B}$ is a basis for V.

0. In V: both vectors are in V because

$$-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.$$

1. Span: If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V, then $y = -\frac{1}{3}(x+z)$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{x}{3}\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3}\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

2. Linearly independent:

$$c_1\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0.$$

Basis for Nul A

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A.

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric vector form}} x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis of Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

1. The vectors span Nul A by construction (every solution to $Ax = 0$ has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)

Basis for Col A

Fact

The *pivot columns* of A always form a basis for Col A.

Warning: I mean the pivot columns of the *original* matrix A, not the row-reduced form. (Row reduction changes the column space.)

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ⇜ pivot columns in rref

So a basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

Why? See slides on linear independence.