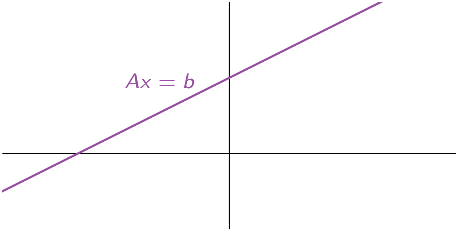
Section 1.5

Solution Sets of Linear Systems

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using spans.



Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Homogeneous Systems

Everything is easier when $b = 0$, so we start with this case.

Definition

A system of linear equations of the form $Ax = 0$ is called **homogeneous.**

These are linear equations where everything to the right of the $=$ is zero. The opposite is:

Definition

A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **nonhomogeneous** or **inhomogeneous.**

A homogeneous system always has the solution $x = 0$. This is called the **trivial solution.** The nonzero solutions are called **nontrivial.**

Observation

$$
\begin{aligned}
Ax = 0 &\text{ has a nontrivial solution} \\
&\iff \text{there is a free variable} \\
&\iff A \text{ has a column with no pivot.}
\end{aligned}
$$

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

We know how to do this: first form an augmented matrix and row reduce.

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

The only solution is the trivial solution $x = 0$.

Observation

Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} \quad x_1 - 3x_2 = 0$$

$$\xrightarrow{\text{parametric form}} \quad \begin{cases} x_1 = 3x_2 \\ x_2 = x_2 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

This last equation is called the **parametric vector form** of the solution.

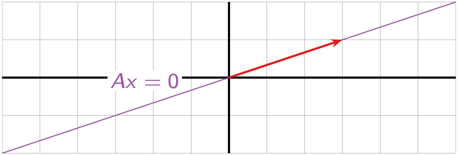
It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in $\mathbf{R}$. The solution set is $\mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.



Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} \quad x_1 - x_2 + 2x_3 = 0$$

$$\xrightarrow{\text{parametric form}} \quad \begin{cases} x_1 = x_2 - 2x_3 \\ x_2 = x_2 \\ x_3 = \qquad\quad x_3 \end{cases}$$

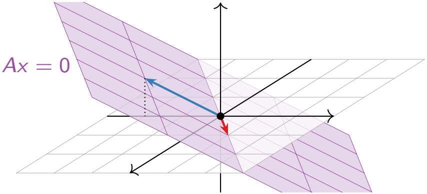
$$\xrightarrow{\text{parametric vector form}} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$

Answer: $\quad \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$



[interactive]

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where $A =$

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \left\{ \begin{aligned} x_1 \qquad - 8x_3 - 7x_4 &= 0 \\ x_2 + 4x_3 + 3x_4 &= 0 \end{aligned} \right.$$

$$\xrightarrow{\text{parametric form}} \left\{ \begin{aligned} x_1 &= 8x_3 + 7x_4 \\ x_2 &= -4x_3 - 3x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \right.$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Homogeneous Systems

Example 2, continued

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}?$$

Answer: $\quad \text{Span}\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$

[not pictured here]

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^4$ ($4 = \#$ variables $= \#$ columns).

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are $x_i, x_j, x_k, \ldots$

Then the solutions to $Ax = 0$ can be written in the form

$$x = x_i v_i + x_j v_j + x_k v_k + \cdots$$

for some vectors $v_i, v_j, v_k, \ldots$ in $\mathbf{R}^n$, and any scalars $x_i, x_j, x_k, \ldots$

The solution set is

$$\mathrm{Span}\{v_i,\ v_j,\ v_k, \ldots\}.$$

The equation above is called the **parametric vector form** of the solution.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Poll

How many solutions can there be to a homogeneous system with more equations than variables?

A. 0

B. 1

C. ∞

The trivial solution is always a solution to a homogeneous system, so answer A is impossible.

This matrix has only one solution to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This matrix has infinitely many solutions to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{equation}} \quad x_1 - 3x_2 = -3$$

$$\xrightarrow{\text{parametric form}} \quad \begin{cases} x_1 = 3x_2 - 3 \\ x_2 = x_2 + 0 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

The only difference from the homogeneous case is the constant vector $p = \binom{-3}{0}$.

Note that p is itself a solution: take $x_2 = 0$.

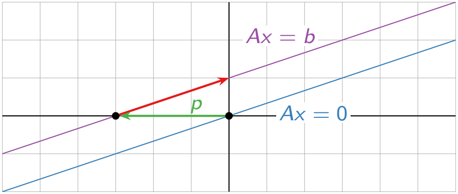
Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ for any x_2 in $\mathbf{R}$.

This is a *translate* of $\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$: it is the parallel line through $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.



It can be written

$$\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2 & 2 & -4 & -2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{equation}} \quad x_1 - x_2 + 2x_3 = 1$$

$$\xrightarrow{\text{parametric form}} \quad \begin{cases} x_1 = x_2 - 2x_3 + 1 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Key Observation

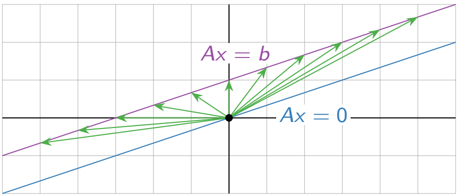
The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

Why? If $Ap = b$ and $Ax = 0$, then

$$A(p + x) = Ap + Ax = b + 0 = b,$$

so $p + x$ is also a solution to $Ax = b$.

We know the solution set of $Ax = 0$ is a span. So the solution set of $Ax = b$ is a *translate* of a span: it is *parallel* to a span. (Or it is empty.)

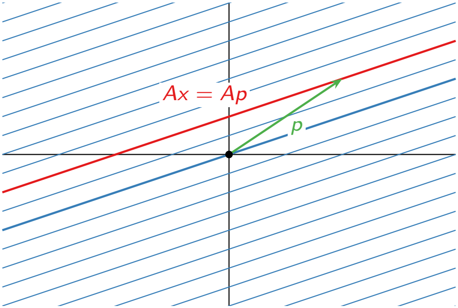


This works for *any* specific solution p: it doesn't have to be the one produced by finding the parametric vector form and setting the free variables all to zero, as we did before.

[interactive]

If we understand the solution set of $Ax = 0$, then we understand the solution set of $Ax = b$ for all b: they are all translates (or empty).

For instance, if $A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$, then the solution sets for varying b look like this:



Which b gives the solution set $Ax = b$ in red in the picture?

Choose p on the red line, and set $b = Ap$. Then p is a specific solution to $Ax = b$, so the solution set of $Ax = b$ is the red line.

Note the cool optical illusion!

[interactive]

For a matrix equation $Ax = b$, you now know how to find which b's are possible, and what the solution set looks like for all b, both using spans.

Solution Sets and Column Spans

Very Important

Let A be an $m \times n$ matrix. There are now two completely different things you know how to describe using spans:

- The **solution set:** for fixed b, this is all x such that $Ax = b$.
  - This is a span if $b = 0$, or a translate of a span in general (if it's consistent).
  - Lives in $\mathbf{R}^n$.
  - Computed by finding the parametric vector form.
- The **column span:** this is all b such that $Ax = b$ is consistent.
  - This is the span of the columns of A.
  - Lives in $\mathbf{R}^m$.

Don't confuse these two geometric objects!

[interactive]