Quiz 7, Discrete Math (15 points), Fall 2016

The quiz is 20 minutes. Show your work and justify your answers where appropriate. If you write the correct answer without sufficient work or justification, you will receive little or no credit.

1. (1 point each) Clearly circle your answer (no justification needed here, and no partial credit given).

   (a) Suppose we use Horner's Algorithm to evaluate $f(x) = 2x^2 - x + 4$ when $x = 2$.
   Originally, we set $S = 2$ (we will keep replacing S until it equals $f(2)$).
   True or false: the next step in Horner's Algorithm is to replace $S = 2$ with $S = 4$. TRUE $\boxed{\text{FALSE}}$

   ($S = 2$ is replaced by $a_1 + Sx = -1 + 2 \cdot 2 = 3$)

   (b) Let A be the set of all functions from $\mathbb{N}$ to $\mathbb{R}$. Define a binary relation $\mathcal{R}$ on A by:

   $$(f, g) \in \mathcal{R} \iff f = \mathcal{O}(g).$$

   Then $\mathcal{R}$ is transitive. $\boxed{\text{TRUE}}$ FALSE

   (Done in homework!)

   (c) If $f, g : \mathbb{N} \to \mathbb{R}$ are given by $f(n) = n^a$ and $g(n) = n^b$ for some real numbers a and b, then $f \asymp g$ if and only if $a = b$. $\boxed{\text{TRUE}}$ FALSE

   From class, we know that if $a < b$ then $n^a \prec n^b$. Similarly if $b < a$ then $n^b \prec n^a$. If $a = b$ then trivially $n^a \asymp n^b$.
   Alternatively, Note $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = \lim_{n\to\infty} \dfrac{n^a}{n^b} = \lim_{n\to\infty} n^{a-b}$. If $a \neq b$ then this is either 0 ($f \prec g$) or ∞ ($g \prec f$), while if $a = b$ then $f = g$ so $f \asymp g$.

   (d) If $f, g : \mathbb{N} \to \mathbb{R}$ are given by $f(n) = n^2$ and $g(n) = 1 + 2 + \cdots + n$, then $f \asymp g$. $\boxed{\text{TRUE}}$ FALSE

   $g(n) = \dfrac{n(n+1)}{2} = \dfrac{n^2}{2} + \dfrac{n}{2}$, so f and g have the same degrees, thus $f \asymp g$.

2. (5 points) Find a particular solution p_n to the recurrence relation

$$a_n = -2a_{n-1} + 2a_{n-2} - n \quad (n \geq 2).$$

(you do not have to worry about initial conditions and you do not need to solve for the homogeneous part q_n)

Solution: We guess $p_n = an + b$.

$$an + b = -2\Big(a(n-1) + b\Big) + 2\Big(a(n-2) + b\Big) - n.$$

$$an + b = \cancel{-2an} + 2a - \cancel{2b} + \cancel{2an} - 4a + \cancel{2b} - n$$

$$an + b = -n - 2a.$$

Setting coefficients equal gives us $a = -1$ and $b = -2a = 2$. So $\boxed{p_n = -n + 2}$.

For this problem, we recall definition 8.2.1 from the textbook, which states:

"Let f and g be functions $\mathbb{N} \to \mathbb{R}$. We say that $f = \mathcal{O}(g)$ if there is an integer n_0 and a positive real number c such that $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$."

3. (6 points) Suppose $f, g : \mathbb{N} \to \mathbb{R}$ are functions whose values $f(n)$ and $g(n)$ are always positive. Use definition 8.2.1 to prove that if $f = \mathcal{O}(g)$, then the function $h(n) = \dfrac{f(n)}{n}$ satisfies $h \prec g$.

   (You cannot use *any* additional theorems or propositions. You must prove $h = \mathcal{O}(g)$ and $g \neq \mathcal{O}(h)$ using definition 8.2.1)

In the solution, we omit all absolute value signs, since all quantities involved are positive for every natural number n.

Solution: Since $f = \mathcal{O}(g)$, there exist $n_0 \in \mathbb{N}$ and $c > 0$ so that $f(n) \leq cg(n)$ for all $n \geq n_0$.

Since $h(n) = \dfrac{f(n)}{n} \leq f(n) \leq cg(n)$ for all $n \geq n_0$, we have $\boxed{h = \mathcal{O}(g)}$.

Suppose for contradiction that $g = \mathcal{O}(h)$, so for some $n_1 \in \mathbb{N}$ and $d > 0$ we have $g(n) \leq dh(n)$ for all $n \geq n_1$. Then for all $n \geq \max\{n_0,\ n_1\}$, we have

$$g(n) \leq d\frac{f(n)}{n} \quad \Longrightarrow \quad ng(n) \leq df(n) \quad \Longrightarrow \quad ng(n) \leq d \cdot cg(n) \quad \Longrightarrow \quad n \leq dc.$$

This is absurd, since it means that dc is greater than every integer! (for example, it fails when $n = \lceil dc + 1 \rceil$). Thus, $\boxed{g \neq \mathcal{O}(f)}$.

We have shown $h \prec g$.