Quiz 6, Discrete Math (15 points), Fall 2016

1. (4 points) Find the sum of the first 20 terms of the arithmetic sequence

$$a_1 = 1, \qquad a_2 = 5, \qquad a_3 = 9, \qquad a_4 = 13, \qquad \ldots.$$

Solution: We use the formula

$$S = n\left(a + \frac{d(n-1)}{2}\right).$$

Here $n = 20$, $a = 1$ and $d = 4$, so

$$S = 20\left(1 + \frac{4 \cdot 19}{2}\right) = 20(1 + 38) = 20(39) = 780.$$

2. (4 points) Solve the recurrence relation

$$a_0 = 2, \quad a_1 = 3, \qquad a_n = 6a_{n-1} - 9a_{n-2} \quad (n \geq 2).$$

Solution: The characteristic polynomial is $x^2 - 6x + 9 = 0$, so $(x-3)^2 = 0$ gives us the repeated root $x = 3$.

$$a_n = c_1 3^n + c_2 n 3^n.$$

Since $a_0 = 2$, we have $2 = c_1 + 0c_2$, so $c_1 = 2$.

Since $a_1 = 3$, we have $3 = c_1 \cdot 3 + c_2 \cdot 1 \cdot 3, \qquad 3 = 6 + 3c_2 \qquad 3c_2 = -3 \qquad c_2 = -1.$

Therefore,

$$a_n = 2 \cdot 3^n - n3^n.$$

3. (7 points) Use induction to prove that for every integer $n \geq 2$, we have

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \; < \; 2 - \frac{1}{n}.$$

Solution: We verify the base case $n = 2$: $\frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4}$ while $2 - \frac{1}{2} = \frac{3}{2} = \frac{6}{4}$, so the inequality holds when $n = 2$.

Suppose the inequality holds for $n = k$, that is, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \; < \; 2 - \frac{1}{k}$.

Using our induction hypothesis, we find

$$\begin{aligned}
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2} &= 2 - \frac{(k+1)^2}{k(k+1)^2} + \frac{k}{k(k+1)^2} \\
&= 2 + \frac{-k^2 - 2k - 1}{k(k+1)^2} + \frac{k}{k(k+1)^2} \\
&= 2 + \frac{-k^2 - k - 1}{k(k+1)^2} \\
&< 2 + \frac{-k^2 - k}{k(k+1)^2} \\
&= 2 + \frac{-k(k+1)}{k(k+1)^2} \\
&= 2 - \frac{1}{k+1}.
\end{aligned}$$

Thus, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$, hence the inequality holds when $n = k + 1$, which is exactly what we needed to show.