Quiz 3, Discrete Math (15 points), Fall 2016

The quiz is 20 minutes. Show your work and justify your answers where appropriate. If you write the correct answer without sufficient work or justification, you will receive little or no credit. If you are asked to prove something is true, provide a rigorous mathematical proof to show it is true. Do not attempt proof-by-paragraph.

1. (1 point each) Clearly circle your answer (no justification needed here, and no partial credit given).

   (a) Suppose $\mathcal{R}$ is a binary relation on a set A, and let $a \in A$. If $\mathcal{R}$ is not reflexive, then $(a, a) \notin \mathcal{R}$.
   TRUE $\boxed{\text{FALSE}}$
   Let $A = \{1, 2\}$ with $\mathcal{R} = \{(1, 1)\}$, and let $a = 1$. Note $\mathcal{R}$ is not reflexive, but $(1, 1) \in \mathcal{R}$.
   Question lifted directly from textbook in section 2.3's true/false section.

   (b) Let $A = \{2, 3, 4\}$, and let $\mathcal{R}$ be the relation on A given by $\mathcal{R} = \{(2, 2), (3, 4), (4, 3), (4, 4)\}$. Then $\mathcal{R}$ is transitive. TRUE $\boxed{\text{FALSE}}$
   In the definition of transitive, take $a = c = 3$ and $b = 4$. Note $(3, 4) \in \mathcal{R}$ and $(4, 3) \in \mathcal{R}$ but $(3, 3) \notin \mathcal{R}$.

   (c) Let $A = \{1, 2, 3, 4, 5, 6\}$. There is an equivalence relation $\mathcal{R}$ on A that satisfies $\bar{2} \cap \bar{6} = \{4\}$.
   TRUE $\boxed{\text{FALSE}}$

   By Proposition 2.4.4, either $\bar{2} \cap \bar{6} = \emptyset$ or $\bar{2} = \bar{6}$. In the latter case, 2 and 6 are elements of $\bar{2} \cap \bar{6}$, so $\bar{2} \cap \bar{6} \neq \{4\}$.

2. (4 points) Suppose $f : A \to B$ and $g : B \to C$ are functions. Prove that if f is one-to-one and g is one-to-one, then $g \circ f$ is one-to-one.

   Solution: Suppose $x_1, x_2 \in A$ and $(g \circ f)(x_1) = (g \circ f)(x_2)$, so $g(f(x_1)) = g(f(x_2))$. Since g is one-to-one,

$$g\Big(f(x_1)\Big) = g\Big(f(x_2)\Big) \implies f(x_1) = f(x_2).$$

   Since f is one-to-one, the fact that $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

   We have shown $(g \circ f)(x_1) = (g \circ f)(x_2) \implies x_1 = x_2$, which means $g \circ f$ is one-to-one.

3. (8 points) Define a binary relation $\mathcal{R}$ on $\mathbb{Z}$ in the following manner:

If $a, b \in \mathbb{Z}$, then $(a, b) \in \mathcal{R}$ if and only if $2a + 5b = 7k$ for some integer k.

(a) (4 points) Is $\mathcal{R}$ reflexive? Either prove it is reflexive, or give a counterexample showing it is not reflexive.

Solution: $\mathcal{R}$ is reflexive. If $a \in \mathbb{Z}$, then $2a + 5a = 7a$, hence $(a, a) \in \mathcal{R}$.

(b) (4 points) Prove that $\mathcal{R}$ is transitive.

Solution: Suppose $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$, so $2a + 5b = 7k$ and $2b + 5c = 7\ell$ for some integers k and ℓ. We add both equations to get

$$(2a + 5b) + (2b + 5c) = 7k + 7\ell$$

$$2a + 7b + 5c = 7k + 7\ell$$

$$2a + 5c = 7k + 7\ell - 7b$$

$$\boxed{2a + 5c = 7(k + \ell - b)}.$$

Since $k + \ell - b$ is an integer, we have $(a, c) \in \mathcal{R}$.