

Quiz 10, Discrete Math (15 points), Fall 2016

The quiz is 20 minutes. Show your work and justify your answers where appropriate. If you write the correct answer without sufficient work or justification, you will receive little or no credit.

1. (1 point) True or false (no partial credit, no justification required):

$$\binom{15}{5} = \binom{14}{4} + \binom{14}{5}. \quad \boxed{\text{TRUE}} \quad \text{FALSE}$$

2. (1 point) True or false (no partial credit, no justification required):

$$\binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \cdots + \binom{10}{9} = 2^{10}. \quad \text{TRUE} \quad \boxed{\text{FALSE}}$$

3. (4 points) Find the coefficient of x^{32} in the binomial expansion of

$$\left(7x^4 - \frac{3}{x^2}\right)^{14}.$$

Solution: We need to solve $x^{32} = (x^4)^{14-k}(x^{-2})^k = x^{56-4k-2k}$ so $6k = 24$, hence $k = 4$. By the Binomial Theorem, the x^{32} term is

$$\binom{14}{4}(7x^4)^{10}\left(-\frac{3}{x^2}\right)^4 = \binom{14}{4}7^{10}(-3)^4x^{40}x^{-8} = \binom{14}{4}7^{10}(-3)^4x^{32},$$

so the coefficient is $\binom{14}{4}7^{10}(-3)^4$.

4. (5 points) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. How many partitions of A consist of exactly 4 parts?

Solution: The possibilities for the number of elements in the parts of the partitions are as follows. For each case, we count the number of partitions. Our answer is the sum of the numbers below.

$$1, 1, 1, 4: \quad \frac{\binom{7}{1}\binom{6}{1}\binom{5}{1}}{3!} \quad \left(\text{or } \frac{\binom{7}{4}\binom{3}{1}\binom{2}{1}}{3!}\right)$$

$$1, 1, 2, 3: \quad \frac{\binom{7}{1}\binom{6}{1}\binom{5}{2}}{2!} \quad \left(\text{or } \frac{\binom{7}{3}\binom{4}{2}\binom{2}{1}}{2!}\right)$$

$$1, 2, 2, 2: \quad \frac{\binom{7}{1}\binom{6}{2}\binom{4}{2}}{3!} \quad \left(\text{or } \frac{\binom{7}{2}\binom{5}{2}\binom{3}{2}}{3!}\right)$$

Our final answer:

$$\frac{\binom{7}{1}\binom{6}{1}\binom{5}{1}}{3!} + \frac{\binom{7}{1}\binom{6}{1}\binom{5}{2}}{2!} + \frac{\binom{7}{1}\binom{6}{2}\binom{4}{2}}{3!}$$

5. (4 points) Find a very simple expression for

$$\binom{n}{0} + 4\binom{n}{1} + 16\binom{n}{2} + \cdots + 4^n\binom{n}{n},$$

and show your answer is correct by using the Binomial Theorem.

Solution: This problem is practically identical to one of our homework problems (7.7 #21(b)).

This sum is

$$\sum_{k=0}^n \binom{n}{k} 4^k = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 4^k = (1+4)^n = 5^n.$$