Probability Comprehensive Exam
Spring 2020

Student Number: ☐

Instructions: Complete 5 of the 9 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8 9

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. If X and Y are integrable random variables and $\mathbb{E}[X \mid Y] = \mathbb{E}X$ a.s., does it follow that X and Y are independent? Prove or provide a counterexample.

2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with

$$\mathbb{P}(X_i = \pm 1) = \frac{1}{2}.$$

Letting $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$, show that for any $L > 0$, there exists $c > 0$ such that for all n sufficiently large,

$$\mathbb{P}(S_i \in [-L, L] \text{ for all } i = 0, \ldots, n) \leq e^{-cn}.$$

3. For $\alpha > 0$, suppose that X is a random variable satisfying

$$\mathbb{P}(|X| \geq t) \leq \frac{1}{t^\alpha} \text{ for all large } t.$$

Show that for any $\beta \in [0, \alpha)$, one has $\mathbb{E}|X|^\beta < \infty$. Is it true for $\beta = \alpha$?

4. Suppose that $X_1, X_2, \ldots$ are random variables that are identically distributed (but not necessarily independent), and have the uniform distribution on $[0, 1]$. Must it be that

$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} \text{ exists a.s.?}$$

Prove or provide a counterexample.

5. Let $X_1, X_2, \ldots$ be i.i.d. uniform $[0, 1]$ random variables and let Y_n be the second mimimum of the X_i's: putting $m_n = \min\{X_1, \ldots, X_n\}$, set

$$Y_n = \min\left(\{X_1, \ldots, X_n\} \setminus \{m_n\}\right).$$

Show that nY_n converges in distribution and compute the density function for the limit.

6. The *Lévy concentration function* of a real valued random variable X is defined as

$$Q(X, t) := \sup_{\lambda \in \mathbf{R}} \mathbb{P}\big\{|X - \lambda| \leq t\big\}, \quad t \geq 0.$$

Show that whenever X and Y are independent, we have

$$Q(X + Y, t) \leq \min\big(Q(X, t), Q(Y, t)\big)$$

for all $t \geq 0$. For simplicity, you may assume that both X and Y have uniformly bounded distribution densities.

7. Let (X_n) be a sequence of i.i.d. non-negative random variables with $\mathbb{E}X_i = 1$. Prove that the sequence of products $\prod_{i=1}^n X_i$ converges almost surely to 0 as $n \to \infty$, unless $X_i = 1$ a.e.

8. Let $(p_n)_{n\geq 1}$ be a sequence of real numbers in $(0, 1]$ such that $\lim_{n\to\infty} np_n = \infty$. Further, for each $n \geq 1$ and for $m = 1, 2, \dots, n$, let X_{nm} be the random variable with

$$\mathbb{P}\big\{X_{nm} = 0\big\} = 1 - p_n, \quad \mathbb{P}\big\{X_{nm} = \sqrt{1/p_n}\big\} = \mathbb{P}\big\{X_{nm} = -\sqrt{1/p_n}\big\} = \frac{p_n}{2}.$$

Assume that for each n, the variables $X_{n1}, \dots, X_{nn}$ are mutually independent. Show that the sequence of random variables $\big(\frac{X_{n1}+\cdots+X_{nn}}{\sqrt{n}}\big)_{n\geq 1}$ converges in distribution to the standard normal.

9. The variables $(X_n)_{n\geq 1}$ are i.i.d. Further, it is known that the sequence of random variables $\big(\frac{X_1+X_2^3+X_3^5+\cdots+X_n^{2n-1}}{\sqrt{n}}\big)$ converges in distribution to the standard normal. Show that X_i's must be the standard sign variables, i.e. $\mathbb{P}\{X_i = 1\} = \mathbb{P}\{X_i = -1\} = 1/2$.