Analysis Comprehensive Exam
Fall 2020

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTES:

- All functions in this exam are real-valued.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.

1. Prove that $E \subseteq \mathbf{R}^d$ is measurable if and only if $|Q| = |Q \cap E|_e + |Q \backslash E|_e$ for every box Q in $\mathbf{R}^d$.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. For a sequence of sets $A_n \in \mathcal{A}$, let
$$B = \limsup_n A_n = \{x \,:\, x \in A_n \text{ for infinitely many } n\}.$$
Suppose that $\inf_n \mu(A_n) = \alpha > 0$.
   (a) Show that $B \in \mathcal{A}$
   (b) Show that $\mu(B) \geq \alpha$.
   (c) Show that the assumption that $\mu(X) < \infty$ is necessary. That is, construct $(X, \mathcal{A}, \mu)$, and measurable sets $A_n \in \mathcal{A}$ with $\mu(A_n) \geq 1$ for all n, but $\limsup_n A_n$ has measure zero.

3. Let S be a closed subspace of $L^1[0,1]$. Suppose that for each $f \in S$, there is some index $p > 1$ such that $f \in L^p[0,1]$. Show that there is a single index $p > 1$ such that $S \subseteq L^p[0,1]$.
   Hint: Consider $A_{n,k} = \big\{f \in S \colon \|f\|_{1+\frac{1}{n}} \leq k\big\}$.

4. Suppose that $f \in \mathrm{AC}[a,b]$ satisfies $f(a) = 0$. Show that
$$\int_a^b |f(x)\, f'(x)|\, dx \;\leq\; \frac{1}{2}\bigg(\int_a^b |f'(x)|\, dx\bigg)^2.$$

5. Let $\{e_n\}_{n\in\mathbf{N}}$ be an orthonormal sequence in $L^2[a,b]$. Prove that $\{e_n\}_{n\in\mathbf{N}}$ is complete in $L^2[a,b]$ if and only if
$$\sum_{n=1}^{\infty} \left|\int_a^x e_n(t)\, dt\right|^2 \;=\; x - a, \qquad \text{for all } x \in [a,b].$$

6. (a) Let $E \subseteq \mathbf{R}^d$ be measurable with $|E| < \infty$. Assume that $\{f_n\}_{n\in\mathbf{N}}$ is a bounded sequence in $L^2(E)$, i.e., $\sup_n \|f_n\|_2 < \infty$. Suppose there exists a function f such that $f_n(x) \to f(x)$ for a.e. $x \in E$. Prove that $\|f - f_n\|_1 \to 0$ as $n \to \infty$.

   (b) Show that the conclusion of part (a) can fail if $|E| = \infty$.

7. Let $f \,:\, [0,1] \to [0,\infty)$ be measurable, and satisfy
$$\int_A f(x)\, dx \leq \sqrt{|A|}$$
for all measureable $A \subseteq [0,1]$. Conclude that $f \in L^p(0,1)$ for all $1 < p < 2$. Show that f need not be in $L^2(0,1)$.

8. Let X, Y, Z be Banach spaces. Suppose that $B\colon X \times Y \to Z$ is bilinear, i.e., $B_f(h) = B(f,h)$ and $B^g(h) = B(h,g)$ are linear functions of h for each $f \in X$ and $g \in Y$. Prove that the following three statements are equivalent.

   (a) $B_f\colon Y \to Z$ and $B^g\colon X \to Z$ are continuous for each $f \in X$ and $g \in Y$.

   (b) There is a constant $C > 0$ such that

$$\|B(f,g)\| \le C\,\|f\|\,\|g\|, \qquad f \in X,\ g \in Y.$$

   (c) B is a continuous mapping of $X \times Y$ into Z (note that B need not be linear on the domain $X \times Y$).