Probability Comprehensive Exam
Fall 2019

Student Number: ☐

Instructions: Complete 5 of the 10 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8 9 10

Write **only on the front side** of the solution pages. A student will pass the exam if 3 problems are worked "almost perfectly" and some progress is made on a fourth problem.

1. First question.

   Show that a random variable X such that

   $$\mathbb{E}[e^{\lambda X}] \leq e^{2|\lambda|^3} \text{ for any } \lambda \in [-1,1]$$

   satisfies $X = 0$ almost surely.

2. Assume $(X_n)_{n\geq 1}$ are iid (independent and identically distributed) random variables on some space $(\Omega, \mathcal{F}, \mathbb{P})$ with common Gumbel cumulative function given by

   $$F(x) = e^{-e^{-x}}, x \in \mathbb{R}.$$

   Show that

   $$\limsup_{n\to\infty}(-X_n - \ln(\ln(n))) = 0.$$

3. Let $(a_i)_{i\geq 1}$ be a sequence of positive integers such that $a_i \in [\lceil 1.01^{i-1}\rceil, \lceil 1.01^i \rceil]$ for all $i \geq 1$. Further, let $(S_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}$, with $S_0 = 0$ and with $S_n = \sum_{i=1}^n X_i$, where $(X_i)_{i\geq 1}$ are mutually independent random variables, with $\mathbb{E}[X_i] = 0$ for all $i \geq 1$, and $|X_i| = a_i$, $i \geq 1$, everywhere on the probability space. Prove that the random walk (S_n) is not recurrent.

4. 1. If X is a random variable such that for two constants $a, b \in \mathbb{R}$, we have $a \leq X \leq b$, show that $\mathrm{var}(X) \leq (b-a)^2/4$ and give an example of such a random variable where equality is attained.
   2. Assume that X is a random variable such that $\mathbb{P}(X \leq a) = 1/2$ and $\mathbb{P}(X \geq b) = 1/2$ for some real numbers a, b, $a < b$. Show that $\mathrm{var}(X) \geq (b-a)^2/4$ and give an example of such a random variable where equality is attained.

5. Let $(S_n)_{n\geq 1}$ be a simple random walk on $\mathbb{Z}$ (with $S_0 = 0$). Compute the probability mass function of the maximum of the random walk on the interval $[2n, 4n]$, i.e. compute the pmf of the variable $\xi := \max_{2n\leq i\leq 4n} S_i$. Represent the pmf as a (polynomial) function of binomial coefficients.

6. Assume $(X_n)_{n\geq 1}$ is a sequence of iid positive random variables. Show that

   $$\frac{X_1 + X_2^2 + \cdots + X_n^n}{n} \xrightarrow[n\to\infty]{a.s.} 1 \text{ if and only if } X_1 = 1 \text{ almost surely.}$$

7. (Modified Polya's urn) Consider the following discrete time process. Before time one, we have one white and one black ball in the urn. At time k ($k \geq 1$), we pick a ball from the urn uniformly at random **with replacement**, and add to the urn a ball of the color opposite to the color of the ball we have picked. Thus, at each step the number of balls in the urn increases by one. Let X_n be the proportion of white balls in the urn right after the n-th step. Show that (X_n) converges to $1/2$ almost surely.

8. Let $b_1, b_2, \ldots$ be a sequence of mutually independent Bernoulli(1/2) random variables, and let m be a fixed positive integer. Define a process $(X_i)_{i\geq 0}$ as follows. Set $X_0 = 1$, and define X_i recursively as

$$X_i = \begin{cases} X_{i-1} + \frac{1}{m}(2b_i - 1), & \text{if } X_{i-1} \geq 1/m; \\ 0, & \text{otherwise.} \end{cases}$$

Show that (X_n) converges to zero almost everywhere.

9. Let $m \geq 2$ be a positive integer, and let $b_1, b_2, \ldots$ be mutually independent Bernoulli(1/2) variables. Consider the following Markov chain (X^n) in $\mathbb{R}^m$. Let X^0 be a fixed 0/1–vector in $\mathbb{R}^m$. Next, given $X^{i-1} = (x_1^{i-1}, x_2^{i-1}, \ldots, x_m^{i-1})$, we set $X^i = (x_1^i, x_2^i, \ldots, x_m^i)$ to be the 0/1–vector such that

$$\sum_{j=1}^{m} 2^{n-j} x_j^i - \sum_{j=1}^{m} 2^{n-j} x_j^{i-1} = \begin{cases} b_i(1 - 2^m), & \text{if } x_1^{i-1} = x_2^{i-1} = \cdots = x_m^{i-1} = 1; \\ b_i, & \text{otherwise.} \end{cases}$$

(Above, "$i-1$", "i" are upper indices, not powers)

1) Prove that the Markov chain (X^n) converges in distribution to the uniform distribution on the set $\{0,1\}^m$.

2) Recall that the mixing time t_{mix} is defined as the smallest integer such that for all $n \geq t_{mix}$ the total variation distance between the distribution of X^n and the stationary distribution is at most $1/4$. Show that the mixing time t_{mix} of (X^n) satisfies $c\,2^{2m} \leq t_{mix} \leq C\,2^{2m}$ for some universal constants $c, C > 0$. You should not use as a blackbox "known" estimates on mixing times, please outline the proof.

10. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ assume we have three random variables X, Y, Z independent and uniform on $[0,1]$. Compute $\mathbb{E}[\min\{X, Y, Z\}]$.