Numerical Analysis Comprehensive Exam
Fall 2017

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Suppose $f(x)$ is a sufficiently smooth real function defined on $(-\infty, \infty)$. Given $f(0) = a_1$, $f'(0) = a_2$, $f(h/2) = a_3$, and $f(h) = a_4$ where $a_1, \cdots, a_4$ are real constants, $h > 0$, find a lowest degree interpolation polynomial $p(x)$ satisfying the same conditions on $[0, h]$ and derive an asymptotic error estimate for the approximation as $h \to 0$.

2. Show that for a sufficiently smooth real function $f(x)$ the Gaussian quadrature with m Gaussian points approximates the integral of $f(x)$ on $[0, h]$,
$$\int_0^h f(x)dx = \sum_{i=1}^m c_i f(x_i) + O(h^{2m+1}),$$
as $h \to 0$, where c_i and $x_i \in [0, h]$ are the Gaussian weights and points respectively. Let $\phi_i(x)$ be the ith Lagrangian basis function for the Lagrangian polynomial interpolation on interpolation points $(x_1, f(x_1)), \cdots, (x_m, f(x_m))$, $1 \le i \le m$. Is it true that for all $1 \le i, j \le m$, $i \ne j$,
$$\int_0^h \phi_i(x)\phi_j(x)dx = 0 \ ?$$
Justify your answer.

3. Consider the initial value problem $\frac{dy}{dt} = f(t, y)$ and $y(0) = y_0$. Show that the improved Euler method (or predictor-corrector method) for solving this problem has a third order local error if f (and y) are sufficiently smooth.

4. Consider the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $(x, t) \in [0, 1] \times [0, T]$, with the initial condition $u(x, 0) = f(x)$ for $x \in [0, 1]$ and boundary conditions $u(0, t) = g_1(t)$ and $u(1, t) = g_2(t)$ for $t \in [0, T]$, where f, g_1 and g_2 are sufficiently smooth functions and $g_1(0) = f(0)$, $g_2(0) = f(1)$. Partition the domain with a uniform grid: $0 = x_0 < x_1 < \cdots < x_M = 1$ and $0 = t_0 < \cdots < t_N = T$ and approximate the equation with an implicit scheme
$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2}$$
where $\Delta t,\ \Delta x$ are the mesh sizes in time and space respectively, $U_i^n \approx u(x_i, t_n)$ and it becomes an equation at the boundary or initial time. Assuming a sufficiently smooth solution, show that $|U_i^n - u(x_i, t_n)| = O(\Delta t + \Delta x^2)$ as $\Delta t,\ \Delta x \to 0$, for any $0 < i < M$, $0 < n \le N$.

5. Consider the system
$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = 0,$$
where
$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Find a stable explicit scheme to solve it. Derive the truncation error and stability condition for the scheme.

6. Let $f(x) = (x-\alpha)^p h(x)$, where h is a sufficiently smooth function, $h(\alpha) \neq 0$, p is a positive integer, and $p \geq 2$. Analyze the rate of convergence of the following modified Newton's method for finding the solution α of $f(x) = 0$, given a sufficiently close initial guess.

$$x_{n+1} = x_n - a\frac{f(x_n)}{f'(x_n)} \equiv g(x_n),$$

where a is a constant. What values of a will make it convergent, and convergent much faster ? what are the convergence rates ?

7. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be a real square matrix with A_{11}, A_{12}, A_{21}, and A_{22} being $k \times k$ matrices.

   (a) Suppose A_{11} is nonsingular. Show that after k steps of Gaussian elimination without pivoting, A_{22} is overwritten by $A_{22} - A_{21}A_{11}^{-1}A_{12}$.

   (b) Suppose $A = A^T$, A_{11} and $-A_{22}$ are positive definite. Show that A is nonsingular and Gaussian elimination in exact arithmetic can be used to find A^{-1}. Describe the procedure of finding A^{-1}.

8. Let $u, v \in \mathbb{R}^n$ and suppose both A and $A + uv^T$ are non-singular.

   (a) Show that $(A + uv^T)^{-1} = A^{-1} - \dfrac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}$.

   (b) Suppose we also have a very fast algorithm to solve $Ax = b$. Describe an algorithm that can solve

   $$(A + uv^T)y = c$$

   very fast too.