Analysis Comprehensive Exam
Spring 2020

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTES:

- All functions in this exam are real-valued.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set A is denoted by χ_A.

1. Let $E \subseteq \mathbf{R}^d$ be measurable, and choose $1 < p < \infty$. Assume that functions $f_n \in L^p(E)$ satisfy $f_n \to f$ a.e. and $\sup \|f_n\|_p < \infty$. Prove that $f \in L^p(E)$, and for each $g \in L^{p'}(E)$ we have that
$$\lim_{n\to\infty} \int_E f_n g = \int_E fg.$$
Does the same result hold if $p = 1$?

2. Compute $\displaystyle\int_0^1 \int_y^1 x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) dx\, dy$

3. Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is absolutely continuous on every compact interval and $f' \in L^2(\mathbf{R})$. Prove that $\displaystyle\sum_{n\in\mathbf{Z}} |f(n+1) - f(n)|^2 < \infty$.

4. (a) Prove that $\varphi(x) = x \ln x$ is convex on $(0, \infty)$.

   (b) Let E be a measurable subset of $\mathbf{R}^d$ such that $|E| < \infty$. Suppose that $f\colon E \to (0, \infty)$ is a measurable function that satisfies $\frac{1}{|E|}\int_E f = 1$. Prove that
$$\frac{1}{|E|} \int_E f(x)\, \ln f(x)\, dx \geq 0.$$

5. Given a set $E \subseteq \mathbf{R}^d$ with $|E|_e < \infty$, show that the following two statements are equivalent. Remark: A *box* in this problem is a set of the form $[a_1, b_1] \times \cdots \times [a_d, b_d]$, and boxes are *nonoverlapping* if they only intersect on their boundaries.

   (a) E is Lebesgue measurable.

   (b) For each $\varepsilon > 0$ we can write $E = (S \cup A) \setminus B$ where S is a union of finitely many nonoverlapping boxes and $|A|_e,\ |B|_e < \varepsilon$.

6. Suppose that $m > n$ and $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is a Lipschitz, but not necessarily linear, function. Prove that $|\mathrm{range}(f)| = 0$.

7. Let E be a measurable subset of $\mathbf{R}^d$ with $0 < |E| < \infty$ and assume that f is an integrable function on E. $f\colon E \to [-\infty, \infty]$ is integrable. Define $g(x) = \int_E |f(t) - x|\, dt$ for $x \in \mathbf{R}$. Prove that g is absolutely continuous on every finite interval $[a, b]$, and $g(x) \to \infty$ as $x \to \pm\infty$.

8. (a) Let X be a Banach space. Prove that a set $S \subseteq X^*$ is bounded if and only if $\sup\{|\mu(x)| : \mu \in S\} < \infty$ for each $x \in X$.

   (b) Let X be a normed linear space. Prove that $S \subseteq X$ is bounded if and only if $\sup\{|\mu(x)| : x \in S\} < \infty$ for each $\mu \in X^*$.