Differential Equations Comprehensive Exam
Spring 2020

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Find all critical points of $\ddot{x}+\sin x=0$. For each of them, demonstrate rigorously if it is a sink, source, saddle, center, or something else in the full nonlinear system.

2. Consider $\ddot{q}+\dot{q}+(q-1)^5=0$. Does $\lim_{t\to\infty} q(t)$ exist? If yes, what is the limit? If not, why?

3. Can you find a periodic solution of the system
$$\begin{cases} \dot{x} &= y-x^5 \\ \dot{y} &= -x-y \end{cases} \quad ?$$
How about
$$\begin{cases} \dot{x} &= y-x^5+(\sin t)^5 \\ \dot{y} &= -x-y+\cos t \end{cases} \quad ?$$
Explain.

4. Consider the equation $\dot{x}=A(t)x$ with $x\in\mathbb{R}^2$ and
$$A(t)=\begin{bmatrix} 5/3-\cos t & b \\ a & 1/3+\sin t \end{bmatrix}$$
and constant a,b. Is there an unbounded solution as $t\to+\infty$?

5. Consider the following initial value problem
$$\begin{cases} u_t+uu_x=0, \;\; x\in\mathbf{R}, t>0, \\ u(x,0)=\dfrac{1}{1+x^2}\in C^1(\mathbf{R}). \end{cases}$$
Find the time when the solution blows up first.

6. Let $D(0,r)$ be the disk on $\mathbf{R^2}$ centered at the origin with radius r with boundary C . Find the function $u(\rho,\theta)$ in polar coordinates so that it is harmonic on $D(0,r)$ and $u(r,\theta)=1+cos^2(\theta)$ on C.

7. Let Ω be an open bounded subset in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$. Assume that $u\in C^{2,1}(\Omega\times(0,\infty))$ solves the following
$$\begin{cases} u_t-\Delta u=f(x)u, \;\; x\in\mathbf{R}^n, \;\; t>0, \\ u(x,t)=0, \;\; x\in\partial\Omega, \;\; t>0, \\ u(x,0)=g(x), \;\; for\ x\in\Omega, \end{cases}$$
where $g(x)\geq 0$, and there is a finite positive number M such that $|f(x)|\leq M$ for all $x\in\Omega$. Prove that $u(x,t)\geq 0$ for all $(x,t)\in(\Omega\times(0,\infty)$.

8. For any finite constant $c > -1$, prove that there is at most one solution $u \in C^2([0,1] \times [0,\infty))$ to the following problem

$$\begin{cases} u_{tt} - a^2 u_{xx} + cu = f(x,t), \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0 \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ for \ x \in (0,1), \end{cases}$$