Differential Equations Comprehensive Exam
Fall 2020

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Does the following system admit a periodic solution?
$$\begin{cases} \dot{x} = x - y - x(x^2+y^2) \\ \dot{y} = y + x - y(x^2+y^2) \end{cases}$$
If yes, what is its stability? If no, why?

2. Consider $\ddot{q} + \sin(q) = 0$. Can you find a point in the phase space that is not in an ω-limit set? Explain. (Here we will use the convention that heteroclinic and homoclinic connections are ω-limit sets).

3. Solve $\ddot{q} + q + \sin(\omega t) = 0$ with arbitrary initial value.

4. Find the stable manifold of each fixed point in the system $\dot{x} = -\sin x$.

5. Assume that $f(x) \in C^1(\mathbf{R})$ is uniformly bounded function with bounded and continuous first order derivatives. Consider the following initial value probem
$$\begin{cases} u_t + uu_x = -2u, \ x \in \mathbf{R}, t > 0, \\ u(x,0) = f(x). \end{cases}$$
Prove that, if there is a point x_0 such that $f'(x_0) < -2$ then the solution $u(x,t)$ of this problem must blow up in finite time. On the other hand, if $f'(x) \geq -2$ for all $x \in \mathbf{R}$, the this problem has a global C^1 solution.

6. Let $D(0,r)$ be the disk on $\mathbf{R^2}$ centered at the origin with radius r with boundary C . Find the function $u(\rho,\theta)$ in polar coordinates so that it is harmonic on $D(0,r)$ and $u(r,\theta) = 2 + cos^2(\theta)$ on C.

7. Let Ω be a ball in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$. Assume that $u \in C^{2,1}(\Omega \times (0,\infty))$ solves the following
$$\begin{cases} u_t - \Delta u - f(x)u = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ u(x,t) = 0, \ x \in \partial\Omega, \ t > 0, \\ u(x,0) = g(x), \ for \ x \in \Omega, \end{cases}$$
where $g(x) \leq 0$, and there is a finite positive number M such that $|f(x)| \leq M$ for all $x \in \Omega$. Prove that $u(x,t) \leq 0$ for all $(x,t) \in (\Omega \times (0,\infty)$.

8. For any finite constant c, prove that there is at most one solution $u \in C^2([0,1] \times [0,\infty))$ to the following problem
$$\begin{cases} u_{tt} - a^2 u_{xx} + cu_t = f(x,t), \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0 \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ for \ x \in (0,1), \end{cases}$$