Differential Equations Comprehensive Exam
Fall 2019

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A student will pass the exam if 3 problems are worked "almost perfectly" and some progress is made on a fourth problem.

1. Find a fundamental matrix for the linear system $\dot{x} = Ax$, where

$$A = \begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix}.$$

2. Consider $\ddot{q} + \dot{q} + q^3 = 0$. Does $\lim_{t\to\infty} q(t)$ exist? If yes, what is the limit? If not, why?

3. Consider the system

$$\begin{cases} \dot{x} &= f(y) \\ \dot{y} &= g(x) + y^k \end{cases}$$

where f and g are $\mathcal{C}^1$ functions and $k \in \mathbb{N}$. Give a sufficient condition for k so that the system contains no periodic solutions.

4. Consider the system

$$\begin{cases} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{cases}$$

where $\sigma, r, b > 0$ are constant parameters. Can you find all fixed points in this system? What is the linear stability of the fixed point $x = y = z = 0$? If it is unstable, does that guarantee the stability of (one of) the other fixed points?

5. Consider the following initial value problem

$$\begin{cases} u_t + uu_x = -2u, \ x \in \mathbf{R}, t > 0, \\ u(x,0) = g(x) \in C^1(\mathbf{R}). \end{cases}$$

where $g(x)$ has uniformly bounded C^1 norm on $\mathbf{R}$.

Determine the sufficient and necessary conditions on $g(x)$ for this problem to have a unique global smooth solution. For any C^1 solution $u(x,t)$ of this problem, prove that

$$\lim_{t\to\infty} \|u(x,t)\|_{L^\infty_x(\mathbf{R})} = 0.$$

6. Let $B(0,1)$ be the unit ball in $\mathbf{R^3}$ centered at the origin. Find a bounded solution to the following Dirichlet problem outside $B(0,1)$

$$\begin{cases} -\Delta u(x) = 0, \ |x| > 1, \\ u(x) = \dfrac{2}{\sqrt{5 + 4x_2}}, \ for \ |x| = 1. \end{cases}$$

7. Assume that $u \in C^{2,1}((0,\pi) \times (0,\infty))$ solves

$$\begin{cases} u_t - u_{xx} = \frac{1}{2}u, \ x \in (0,\pi), \ t > 0, \\ u(0,t) = u(\pi,t) = 0, \ t > 0, \\ u(x,0) = f(x), \ for \ x \in (0,\pi), \end{cases}$$

where $f(x) \in C_0^\infty(0,\pi)$, that is $f(x)$ has compact support in $(0,\pi)$. Prove that

$$\lim_{t\to\infty} \|u(x,t)\|_{L^2_x([0,\pi])} = 0.$$

8. For any finite constant $c \geq 0$, prove that there is at most one solution $u \in C^2([0,1] \times [0,\infty))$ to the following problem

$$\begin{cases} u_{tt} - a^2 u_{xx} + cu = f(x,t), \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0 \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ for \ x \in (0,1), \end{cases}$$