Differential Equations Comprehensive Exam
Fall 2017

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let u be the entropy solution of

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad \text{in } (0, +\infty) \times \mathbb{R},$$

with the initial condition

$$u(0, x) = \begin{cases} 1 & x < -1 \\ x^2 & -1 \le x \le 1 \\ 1 & x > 1. \end{cases}$$

Let t_0 be the time when the first shock will appear. Find the solution u for points $(t, x) \in [0, t_0] \times \mathbb{R}$. Write the ODE that describes the trajectory of the shock. (You don't have to solve it.)

2. Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ and let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$-\sum_{i=1}^{n-1} u_{x_i x_i} \le 0 \quad \text{in } \Omega.$$

Prove that

$$\max_{x \in \overline{\Omega}} u \le \max_{x \in \partial\Omega} u.$$

It it true that if there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{x \in \overline{\Omega}} u$ then u must be constant in Ω?

3. Let Ω be a bounded domain with smooth boundary and let $u \in C^2([0, +\infty) \times \overline{\Omega})$ be a solution of

$$\begin{cases} u_{tt} - \Delta u + \alpha u + \beta u^3 \quad \text{in } (0, +\infty) \times \Omega \\ u(0, x) = f(x), \quad u_t(0, x) = g(x) \quad \text{in } \Omega \\ u(t, x) = 0 \quad (t, x) \in (0, +\infty) \times \partial\Omega, \end{cases}$$

where $\alpha > 0$ and $\beta \ge 0$ are fixed constants. Show that there exists a constant $C \ge 0$, depending on the initial conditions, such that

$$E(t) = \int_\Omega \left[|u_t(t, x)|^2 + |\nabla u(t, x)|^2 + |u(t, x)|^2\right] dx \le C \quad \text{for all } t > 0.$$

4. Let u be the solution of the heat equation

$$\begin{cases} u_t = u_{xx} \quad \text{in } (0, +\infty) \times \mathbb{R} \\ u(0, x) = f(x) \quad x \in \mathbb{R}, \end{cases}$$

where f is continuous and has compact support. Show that there is a constant C independent of t and u such that

$$\max_{x \in \mathbb{R}} |u_x(t, x)| \le C t^{-\frac{3}{4}} \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{\frac{1}{2}} \quad \text{for all } t > 0.$$

5. Consider the differential equation

$$\ddot{x}(t) = f(t)x(t)$$

with f continuous in $[0, \infty)$ and

$$\int_0^\infty t|f(t)|dt < \infty.$$

Show that it admits a unique solution in $[0, \infty)$ such that

$$\lim_{t\to\infty} x(t) = 1 \qquad\qquad \lim_{t\to\infty} \dot{x}(t) = 0$$

Hint: show first that the solution must satisfy $x(t) = 1 + \int_t^\infty (t - s)f(s)x(s)ds$.

6. Let $y_1(t)$ and $y_2(t)$ be two independent solutions of

$$\ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) = 0$$

Assume that there exist t_1 and t_2 such that $y_1(t_1) = y_1(t_2) = 0$ and $y_1(t) \neq 0$ for $t \in (t_1, t_2)$. Show that there exists a unique $\bar{t} \in (t_1, t_2)$ such that $y_2(\bar{t}) = 0$.

7. Consider the differential equation

$$y^{(n)}(t) + \sum_{k=n-1}^{0} a_k(t)y^{(k)}(t) = 0$$

where the $a_k(t)$ are continuous in (a, b) for some $-\infty < a < b < \infty$. Let $y(t) \not\equiv 0$ be a solution defined in (a, b). Show that if $[c, d] \subset (a, b)$ then $y(t)$ has at most finitely many roots in $[c, d]$. Is the same true in (a, b)?

8. Consider the system of equations

$$\begin{aligned} \dot{x} &= x - y - (3x + y)(x^2 + y^2) + 2x(x^2 + y^2)^2 \\ \dot{y} &= x + y - (3y - x)(x^2 + y^2) + 2y(x^2 + y^2)^2 - ax^2y \end{aligned}$$

Show that for a small enough this system admits at least 2 periodic orbits.