Analysis Comprehensive Exam
Fall 2019

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A student will pass the exam if 3 problems are worked "almost perfectly" and some progress is made on a fourth problem.

1. Let $E \subset \mathbb{R}^m$ be a set of finite measure, and $f : E \to (0, \infty)$ be a measurable function. Suppose that for some $A, B > 0$ we have,

$$A \le f(x) \le B \qquad \text{for a.e. } x \in E.$$

Prove that

$$\lim_{p \to 0+} \int_E \frac{f^p - 1}{p} dx = \int_E (\log f)\, dx.$$

(b) Conclude that

$$\lim_{p \to 0+} \left(\frac{1}{|E|} \int_E f^p dx \right)^{1/p} = \exp\left(\frac{1}{|E|} \int_E (\log f)\, dx \right).$$

(Hint: You may want to use a Taylor approximation of $\log(1 + pt)$ over a bounded interval)

2. Let $\{r_j\}_{j=1}^{\infty}$ be an enumeration of the rational numbers in $(-\infty, \infty)$. Define a set function μ on all subsets S of $(-\infty, \infty)$ by

$$\mu(S) = \sum_{j : r_j \in S} \frac{1}{j^2}.$$

Let $\{p_j\}_{j=1}^{\infty}$ be an enumeration of the prime numbers. Define a set function ν on all subsets S of $(-\infty, \infty)$ by

$$\nu(S) = \sum_{j : p_j \in S} \frac{(-1)^j}{2^j}.$$

(a) Show that μ, ν are countably additive (signed) measures on the $\sigma-$algebra Σ of all subsets of the real line.
(b) Show that ν is absolutely continuous with respect to μ but μ is not absolutely continuous with respect to ν.
(c) Find the Lebesgue (or if you prefer Radon-Nikodym) decomposition of μ with respect to ν, so that

$$\mu = \omega + \sigma$$

where σ is singularly continuous with respect to ν, while for some function f,

$$\omega(A) = \int_A f \, d\nu.$$

Also find f.
(d) Find the Hahn-Jordan Decomposition of ν, that is

$$\nu = \nu_+ - \nu_-$$

where ν_- and ν_+ are positive measures.

3. Let $0 < \beta < 1$.
(a) Construct a measurable set E in $[-1, 1]$ such that

$$\limsup_{\delta \to 0+} \frac{|E \cap [-\delta, \delta]|}{2\delta} = \beta$$

but

$$\liminf_{\delta \to 0+} \frac{|E \cap [-\delta, \delta]|}{2\delta} = 0.$$

(b) What does Lebesgue's differentiation theorem say about

$$\lim_{\delta \to 0+} \frac{|E \cap [x - \delta, x + \delta]|}{2\delta}$$

for a.e. $x \in E$? Just state your answer for (b), do not prove it.

4. Let $f : [0, 1] \to [0, \infty)$ be a function which satisfies

$$\int_0^1 e^{sf(x)} dx \le e^{s^2}, \qquad \forall s > 0.$$

i. Prove that for every $t > 0$ we have

$$|\{x \in [0, 1] : f(x) > t\}| < e^{-t^2/4}.$$

ii. Prove that f is integrable, that is, prove that

$$\int_0^1 f(x) dx < \infty.$$

5. Prove that there exists a function $f : [0, 1] :\to [0, \infty)$ which is absolutely continuous, strictly increasing (that is, $f(t) > f(s)$ whenever $t > s$), and such that

$$|\{t \in [0, 1] : f'(t) = 0\}| > 0.$$

6. (a) Show that the set of all irrational numbers is a G_δ set.
(b) Show that the set $\mathbb{Q}$ of all rational numbers is an F_σ set, but is not a G_δ set.

7. For $f \in L^1[0, 1]$ denote

$$\widehat{f}(n) = \int_0^1 f(t) e^{-2\pi i n t} dt.$$

Prove that for every $f \in L^1[0, 1]$ we have $\widehat{f}(n) \to 0$ as $|n| \to \infty$.

(Remark: You are asked here to prove the result. Stating the fact that this is a known result will not be considered as a proof).

8. Let H be a Hilbert space, and let $\{e_n\}_{n\in\mathbb{N}} \subset H$ be a sequence of elements in H. Assume that for every $f \in H$ we have

$$\sum_n |\langle f, e_n\rangle|^2 < \infty.$$

i. Prove that there exists $B > 0$ such that

$$\sum_n |\langle f, e_n\rangle|^2 \leq B\|f\|_H^2.$$

ii. Determine whether the following statement is true or false, and prove or disprove it accordingly: Under the above conditions we have

$$\sup_{f\in H} \sum_{n=N}^{\infty} |\langle f, e_n\rangle|^2 \to 0 \qquad \text{as } N \to \infty.$$