Numerical Analysis Comprehensive Exam
Spring 2021

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. (a) Consider a variation of Newton's method in which only one derivative is needed, that is
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$
Does it have quadratic convergence in this case ? Why ? If it converges linearly, what's the convergence rate ?

   (b) Describe and explain the bisection method in detail. Suppose that the bisection method is started with the interval $[10, 34]$. How many steps should be taken to compute a root with an accuracy of 10^{-12}?

2. Let $u(t)$ be a sufficiently smooth function. Let $0 = t_0 < t_1 < t_2 < \cdots$ be a uniform partition, $\Delta t = t_{n+1} - t_n$. Denote $u^n = u(t_n)$. Find a second order approximation to $\frac{du}{dt}(t_{n+1})$ using values of u^{n+1}, u^n and u^{n-1}, and justify your answer.

3. Consider solving an ODE problem $y_t = f(t, y)$.

   (a) What is backward Euler method ? Explicitly write the formula and explain.

   (b) Compute the order of accuracy for backward Euler method, e.g., by considering the local truncation error computation.

   (c) What is the absolute stability region for Backward Euler method ? Consider the Cauchy Problem $y' = \lambda y$, $y(0) = 1$.

4. (a) Solve the Burgers' equation $u_t + (\frac{1}{2}u^2)_x = 0$ with an initial data
$$u(x,0) = \begin{cases} 2, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$
That is, sketch the characteristics and shock paths in the $x - t$ plane.

   (b) Design an upwind scheme for this equation and explain why this is numerically important. Does the numerical solution converge to the solution given in (a) ?

5. Consider the two dimensional scheme to solve $u_t = u_{xx} + u_{yy}$

   (a) Write an Alternating Direction Implicit (ADI) scheme to solve this problem, e.g. Peaceman-Rachford, etc. What is the idea of ADI schemes ?

   (b) What is the error for the scheme you choose ?

6. (a) Compute the matrix norms $\|A\|_1$, $\|A\|_\infty$, $\|A\|_F$ and $\|A\|_2$ for $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Show the details of the computation.

   (b) Show that for any given norm, the condition number has following relations $\kappa(AB) \leq \kappa(A)\kappa(B)$ and that $\kappa(\alpha A) = \kappa(A)$ for all nonzero α.

   (c) Compute the number of arithmetic operations for AB, when $A \in \mathrm{R}^{n\times n}$ is lower triangular and $B \in \mathrm{R}^{n\times n}$ is upper triangular.

7. (a) Compute LU factorization of $A = \begin{pmatrix} 2 & 6 \\ 4 & -2 \end{pmatrix}$ with partial pivoting. That is to find the Lower L and Upper Triangular matrix U. Clearly indicate A, P_k, L_k, U_k for each iterations, L and U.

   (b) What is Cholesky factorization, when can it be used, and what is numerical advantage of using it for solving $Ax = b$? What is a difference compared to using QR factorization ? Explain in details.