Numerical Analysis Comprehensive Exam
Fall 2019

Student Number: [ ]

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A student will pass the exam if 3 problems are worked "almost perfectly" and some progress is made on a fourth problem.

1. Derive the two-point Gaussian quadrature formulas for

$$I = \int_0^1 x f(x) dx \approx \sum_{j=1}^{n} A_j f(x_j)$$

with weight function $w(x) = x$.

2. Let $f(x, y)$ be a sufficiently smooth function. Suppose the values of f are only known at $(0, 0)$, $(0, h)$, (h, h) and $(h, 0)$ as A, B, C and D respectively, $h > 0$.

   (a) Find a numerical approximation of $f(x, y)$, $0 \leq x, y \leq 1$, so that the approximation error is $O(h^2)$ as $h \to 0$ (known as the second order approximation).

   (b) Find a second order numerical approximation of $\frac{\partial f}{\partial x}(h/2, h/2)$, and justify the order of the approximation error.

3. Consider a second order differential equation with initial values given by

$$y'' + 101y' + 100y = 0, \quad y(0) = 1, \quad y'(0) = 98.$$

   (a) Write the equation in the form of first order system, and find its exact solution.

   (b) Give the explicit forward Euler scheme for the first order system, what choice of the step size will guarantee the absolute stability of the scheme.

4. Consider the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $(x, t) \in [0, 1] \times [0, T]$, with the initial condition $u(x, 0) = f(x)$ for $x \in [0, 1]$ and boundary conditions $u(0, t) = g_1(t)$ and $u(1, t) = g_2(t)$ for $t \in [0, T]$, where f, g_1 and g_2 are sufficiently smooth functions and $g_1(0) = f(0)$, $g_2(0) = f(1)$. Partition the domain with a uniform grid: $0 = x_0 < x_1 < \cdots < x_M = 1$ and $0 = t_0 < \cdots < t_N = T$ and approximate the equation with an implicit scheme

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2}$$

where Δt, Δx are the mesh sizes in time and space respectively, $U_i^n \approx u(x_i, t_n)$ and it becomes an equation at the boundary or initial time. Assuming a sufficiently smooth solution, show that $|U_i^n - u(x_i, t_n)| = O(\Delta t + \Delta x^2)$ as $\Delta t,\ \Delta x \to 0$, for any $0 < i < M$, $0 < n \leq N$.

5. Study the numerical stability of the following scheme.

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{1}{2}\{\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2} + \frac{U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}}{\Delta x^2}\} + U_i^n.$$

6. Consider a $n \times 3$ matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 10^{-9} & 0 & 0 \\ 0 & 10^{-9} & 0 \\ 0 & 0 & 10^{-9} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

where $n > 3$, and a vector $b \in \mathbb{R}^n$.

(a) If x is the least squares solution of $Ax = b$. Prove that $Ax = b_1$, where b_1 is the orthogonal projection of b onto the range of A, $\mathcal{R}(A)$.

(b) Assuming that the machine precision is 10^{-16}, can one compute the least squares solution x by the normal equation approach? If your answer is yes, describe the steps and formulas to compute x. If your answer is no, explain why and give a workable approach to compute x.

7. A is a 5-band symmetric $n \times n$ matrix

$$A = \begin{bmatrix} 10 & b_1 & c_1 & 0 & 0 & \cdots & 0 \\ b_1 & 10 & b_2 & c_2 & 0 & \cdots & 0 \\ c_1 & b_2 & 10 & b_3 & c_3 & \cdots & 0 \\ 0 & c_2 & b_3 & 10 & b_4 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & c_{n-2} & b_{n-1} & 10 \end{bmatrix}.$$

Its entries b_i and c_i are random numbers following uniform distributions in intervals $(0, 2)$ and $(-2, 0)$ respectively.

(a) Prove that A is invertible.

(b) Consider solving $Ax = b$ for an arbitrary vector $b \in \mathbb{R}^n$ by Jacobi iterations. Is Jacobi iteration convergent? If your answer is yes, give the error estimate. If not, explain why. You must justify your answer.