Topology Comprehensive Exam
Spring 2021

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $a : S^n \to S^n$ be the antipodal map, which is given by $a(p) = -p$.

    1. Suppose that α is a smooth form on S^n satisfying $a^*\alpha = \alpha$. Prove that if α is exact, then there is a smooth form β on S^n such that $d\beta = \alpha$ and $a^*\beta = \beta$.
    2. Using the fact that the de Rham cohomology $H^k_{DR}(S^n)$ is trivial for $0 < k < n$, show that $H^k_{DR}(\mathbf{R}P^n) = 0$ for $0 < k < n$. You map assume without proof that the quotient map $p : S^n \to \mathbf{R}P^n$ that identifies antipodal points is smooth local diffeomorphism.

2. Find two covering spaces of $S^1 \vee S^1$ that have degree 4, one regular and one not. Prove that the subgroups of $\pi_1(S^1 \vee S^1)$ corresponding to the different degree 4 covers of $S^1 \vee S^1$ are all isomorphic.

3. Let M be a compact manifold without boundary. Let $\Delta = \{(x, x) \in M \times M : x \in M\}$ be the diagonal in $M \times M$. Prove that there is no compact submanifold W in $M \times M$ with $\partial W = \Delta$.
Hint: consider the intersection number of two submanifolds of $M \times M$.

4. Let A and B be one dimensional submanifolds of $\mathbf{R}^3$. Show that for any given $\epsilon > 0$ there is some vector $v \in \mathbf{R}^3$ with length less than ϵ such that $A + v$ is disjoint from B. Here $A + v = \{x + v : x \in A\}$ is the translate of A by v.

5. Compute the fundamental group of the torus. Show that every finite degree covering spaces of the torus is homeomorphic to the torus. Is it true that if a topological space X has the torus as a covering space then X is homeomorphic to a torus? If yes, give a proof, and if not, give a counterexample.

6. Consider the following vector fields in $\mathbf{R}^2$
$$v = y\frac{\partial}{\partial x} - (x+1)\frac{\partial}{\partial y} \text{ and } w = y\frac{\partial}{\partial x} - (x-1)\frac{\partial}{\partial y}$$
and the 1–form
$$\alpha = x\,dy - y\,dx$$

    1. Compute $[v, w]$.
    2. Compute $\mathcal{L}_v\alpha$ where $\mathcal{L}_v$ is the Lie derivative in the direction of v.

7. Let $f : M \to N$ be a degree 1 map between compact oriented manifolds. Show that $f_* : \pi_1(M, x) \to \pi_1(N, f(x))$ is surjective.
Hint: You may use the fact that if $p : \widetilde{X} \to X$ is a k-fold covering map between oriented k–manifolds, then p has degree k.

8. Let X be $\mathbf{R}^3$ minus n lines through the origin. Show that X is homotopy equivalent to $\mathbf{R}^2$ minus a finite set of points. Compute the fundamental group of X.