# Topology Comprehensive Exam
# Spring 2020

**Student Number:** ☐

*Instructions:* Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Construct a topological space $X$ with fundamental group $\mathbf{Z}/3\mathbf{Z}$. Show that any map from $X$ to $S^1$ must be null-homotopic.

2. Let $M$ be a manifold. Define a 1-form $\lambda$ on $T^*M$ as follows. Let $\pi : T^*M \to M$ be the projection map. If $\eta \in T^*M$ and $v \in T_\eta(T^*M)$ then set $\lambda_\eta(v) = \eta(d\pi_\eta(v))$. If $\alpha$ is a 1-form on $M$, then explain how $\alpha$ is a map from $M$ to $T^*M$ and show that $\alpha^*\lambda = \alpha$.

3. For which values of $a > 0$ does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a^2$ transversely in $\mathbf{R}^3$? Justify your answer.

4. Let $M$ be a compact manifold with smooth vector field $X$ and resulting 1-parameter subgroup $H_t^X \leqslant \mathrm{Diff}(M)$; $H_t^X$ is also called a the flow of $X$. Let $f \in \mathrm{Diff}(M)$ and let $f_*(X)$ be the pushforward of $X$. Show that
$$H_t^{f^*(X)} = f \circ H_t^X \circ f^{-1}.$$

5. Show that the set of real $2 \times 2$ matrices of rank 1 is a 3-dimensional submanifold of the space $M_2(\mathbf{R})$ of all $2 \times 2$ matrices.

6. Let $X$ be the quotient space obtained by identifying two distinct points in $S^2$. Compute the fundamental group of $X$.

7. Say that a covering space is abelian if it is a connected, regular cover whose deck transformations form an abelian group. Show that every connected CW-complex $X$ has a universal abelian cover—that is, an abelian cover that covers all other abelian covers—and that this universal cover is unique up to isomorphism of covering spaces.

8. Recall that a map $f : X \to R$ defined on $X \subset \mathbf{R}^n$ is smooth if for each $x$ in $X$ there is a neighborhood $U_x$ and a smooth function $f_x : U_x \to \mathbf{R}$ such that $f = f_x$ on $X \cap U_x$. Show that if $f : X \to \mathbf{R}$ is smooth then there is a smooth extension $F : \mathbf{R}^n \to \mathbf{R}$.