Topology Comprehensive Exam
Fall 2019

Student Number: ☐

Instructions: Complete 5 of the 10 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8 9 10

Write **only on the front side** of the solution pages. A student will pass the exam if 3 problems are worked "almost perfectly" and some progress is made on a fourth problem.

1. Let $\mathcal{L}_v\omega$ denote the Lie derivative of the k-form ω in the direction of the vector field v. Show that for any smooth function we have

$$\mathcal{L}_v f\omega = df(v)\omega + f\mathcal{L}_v\omega.$$

2. Prove that any continuous map of the unit square to itself must have a fixed point. You do not need to compute any fundamental groups that you use in the argument.

3. Compute the fundamental group of $Y = \mathbf{R}P^2 \vee \mathbf{R}P^2$. Describe explicitly the universal cover of Y, prove it is the universal cover, and draw a picture of it.

4. Let $F : M \times [0,1] \to N$ be a smooth homotopy. For each $t \in [0,1]$, let $f_t : M \to N$ be the map $p \to F(p,t)$. Assume that f_0 is an immersion and M is compact. Show that there is an $\epsilon > 0$ so that f_t is an immersion for $t < \epsilon$.

5. Show that any continuous map from a manifold of dimension k to S^n is null-homotopic if $k < n$.

6. Recall that $\mathbf{R}P^n$ is the set of lines (through the origin) in $\mathbf{R}^{n+1}$. Show that $\mathbf{R}P^n$ is a compact smooth manifold (you may assume without proof that $\mathbf{R}P^n$ is Hausdorff and second countable).

7. Let F_k denote the free group of rank k. Determine whether or not there exist subgroups of F_2 of the following forms:

   (a) a normal subgroup isomorphic to F_3

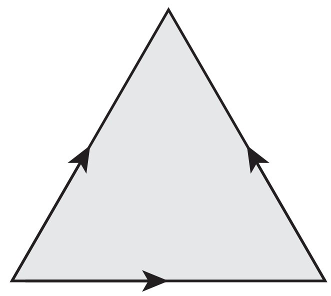
   (b) a non-normal subgroup isomorphic to F_3

   In each case, if there is an example, give explicit generators for the subgroup. If there is no example, explain why not. You should use covering spaces for your justifications.

8. Compute the fundamental group of the space obtained from a torus by identifying 3 of its points to a single point.

9. Let M be a compact manifold with boundary. Show there is no retraction of M to ∂M, that is show there is no smooth map $f : M \to \partial M$ such that $f(p) = p$ for $p \in \partial M$.

10. Consider the space X obtained from a 2-simplex by the identifying the 3 edges to a single edge so that the arrows shown in the figure match.



1. Show that X is simply connected.
2. Show that X is contractible.

You may use the fact that if we glue a 2-cell to a cell complex Y with two different, but homotopic attaching maps f and g, then the resulting complexes $Y \cup_f e^2$ and $Y \cup_g e^2$ are homotopy equivalent.