Analysis Comprehensive Exam
Fall 2017

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let ν be a signed Borel measure on $\mathbf{R}$. Prove that

$$|\nu|(E) = \sup\left\{\left|\int_E f\,d\nu\right| : |f| \le 1\right\}.$$

2. Assume that $1 \le p < q \le \infty$ and $f \in L^p(\mathbf{R}^d) \cap L^q(\mathbf{R}^d)$. Show that $f \in L^r(\mathbf{R}^d)$ for all $p \le r \le q$.

3. Assume that $f\colon \mathbf{R} \to \mathbf{R}$ is monotone increasing, and that we have $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$. Prove that f is absolutely continuous on every finite interval $[a,b]$ if and only if

$$\int_{-\infty}^{\infty} f'(x)\,dx = 1.$$

4. Let ϕ be a function in $C^1(\mathbf{R}^d)$ with compact support. Show that for any $f \in L^1(\mathbf{R}^d)$ the function

$$\phi \star f(x) := \int \phi(x-y)f(y)dy$$

is differentiable.

5. The two parts of this problem are not related.

(a) Exhibit a set $E \subseteq [0,1]$ that is meager yet has measure $|E| = 1$.

(b) Suppose that f is an infinitely differentiable function on $\mathbf{R}$ such that for each $x \in \mathbf{R}$ there exists some integer $n_x \ge 0$ so that $f^{(n_x)}(x) = 0$. Prove that there exists some open interval (a,b) and some polynomial p such that $f(x) = p(x)$ for all $x \in (a,b)$.

6. Minimize

$$\int_{\mathbf{R}} x^2 f(x)\,dx$$

among all measurable functions with $0 \le f(x) \le A$ and $\int_{\mathbf{R}} f(x)dx = B$.

7. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in a Banach space X. Fix $1 \le p \le \infty$ and let p' the dual index to p. Given a linear functional μ in the dual space X^*, set $T(\mu) = (\mu(x_n))_{n\in\mathbf{N}}$, and suppose that $T(\mu) \in \ell^{p'}$ for every $\mu \in X^*$. Prove that T is a bounded mapping of X^* into $\ell^{p'}$.

Turn Over $\to$.

8. Let $f \in L^1(\mathbf{R}^2)$ be a function such that $f(R^{-1}x) = f(x)$ for all $x \in \mathbf{R}^2$, where

$$R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} .$$

Show that if α is an irrational multiple of 2π then for a.e. $x \in \mathbf{R}^2$,

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(R_\theta x)\, d\theta$$

where

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} .$$