Algebra Comprehensive Exam
Fall 2019

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A student will pass the exam if 3 problems are worked "almost perfectly" and some progress is made on a fourth problem.

1. Let A and B be $n \times n$ matrices over a field K such that $A^2 = A$ and $B^2 = B$. Prove that A and B are similar if and only if they have the same rank.

2. If A and B are normal subgroups of a group G such that G/A and G/B are both abelian, prove that $G/(A \cap B)$ is abelian.

3. Let R be a principal ideal domain, let M be a torsion R-module, and let p be a prime in R. Prove that if $pm = 0$ for some nonzero $m \in M$, then the annihilator $Ann(M)$ is a subset of the ideal $\langle p \rangle$.
   Recall that $Ann(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$.

4. Show that in a finite field every element is a sum of two perfect squares. (0 counts as a perfect square.)

5. (a) Show that every prime ideal in a principal domain is maximal.
   (b) Let R be a ring with a unique maximal ideal M. Show that an element of R is invertible if and only if it is not in M.

6. Let p be a prime number. Find two non-isomorphic groups of order $2p$. Show that, up to isomorphism, there are only two groups of order $2p$.

7. Let R be a commutative ring. A polynomial over R is called *primitive* if its coefficients generate R. If $f, g \in R[x]$, show that $f \cdot g$ is primitive if and only if both f and g are primitive. (hint: consider a maximal ideal containing the coefficients of $f \cdot g$).

8. Let ζ be a primitive 11-th root of unity. Use the Galois correspondence to determine the degrees of $\alpha = \zeta^3 + \zeta^8 + 6$ and of $\beta = \zeta^2 + \zeta^3$ over $\mathbf{Q}$.