Algebra Comprehensive Exam
Fall 2020

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let A be a square $n \times n$ matrix with complex entries. Show that A is nilpotent if and only if A and $4A$ are similar.

2. Show that every finite group of order > 2 has an automorphism that is not the idenity map.

3. Let G be a group of order 105. Show that G is not simple.

4. Determine the number of non-isomorphic abelian groups of order 3600.

5. Let R be an integral domain satisfying the ascending chain condition: given an increasing sequence of ideals $I_1 \subseteq I_2 \subseteq I_3 \ldots$ there exists $k \in \mathbb{N}$ such that $I_k = I_{k+1} = I_{k+2} = \cdots$. Suppose that for any $a \in R$ there exists $b \in R$ such that $b^2 = a$. Show that R is a field.

6. Let R be a commutative ring with identity. Suppose that for every $r \in R$ there is an integer $n > 1$ such that $r^n = r$ (n may depend on r). Show that every prime ideal in R is maximal.

7. How many roots does the polynomial $x^{2020} - 1$ have in the finite field $\mathbb{F}_{101}$?

8. Let K be the splitting field of $x^3 - 5$ over $\mathbb{Q}$. What is the order of Galois group of K over $\mathbb{Q}$? Is the Galois group abelian?