Algebra Comprehensive Exam
Fall 2017

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let x and y be two elements of order 2 in a finite group G. Prove that $\langle x, y\rangle$ is either abelian or is isomorphic to a dihedral group.

2. Find a factorization of
$$f = 6x^4 - 4x^3 + 24x^2 - 4x - 8$$
into prime elements of $\mathbf{Z}[x]$.

3. Let A and B be finitely generated abelian groups such that $A \times A \cong B \times B$. Prove that $A \cong B$.

4. Find all primitive elements in the field extension $\mathbf{Q}(\sqrt{2}, \sqrt{3})/\mathbf{Q}$. Justify your answer.

5. Let V be an n-dimensional real vector space with a non-degenerate quadratic form q. Prove that there exists a nonzero vector $v \in V$ such that $q(v) = 0$ if and only if q can be written as $x_1x_2 + \sum_{i,j=3}^{n} a_{ij}x_ix_j$ for some choice of basis.

6. Let p be a prime number, and let G be any p-subgroup of $\mathrm{GL}_n(\mathbf{F}_p)$ for some $n \geq 1$. Prove that there is a nonzero vector $v \in \mathbf{F}_p^n$ such that $gv = v$ for all $g \in G$, with respect to the natural action of $\mathrm{GL}_n(\mathbf{F}_p)$ on $\mathbf{F}_p^n$.

7. Let $R = \mathbf{Q}[x, y]$. Is R an Euclidean domain? Is R a unique factorization domain?

8. Show that $f(x) = x^3 - 3x - 1$ is an irreducible element of $\mathbf{Z}[x]$. Compute the Galois group of the splitting field of f over $\mathbf{Q}$ and over $\mathbf{R}$.