Topology Comprehensive Exam
Spring 2026

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let M and N be compact manifolds without boundary of dimension $k > 0$ and $n-k > 0$, respectively. For any point $x \in N$ let $i_x : M \to M \times N$ be the map $i_x(p) = (p, x)$.

   (a) Show that i_x is not homotopic to a constant map. Hint: use intersection numbers.

   (b) Show that any smooth map from M into S^n is homotopic to a constant map. (Hence S^n is not a non-trivial product.)

2. Let F_n be a nontrivial free group with n generators. Prove that every index k subgroup of F_n is a free group of rank $1+k(n-1)$. Hint: Use a familiar space X whose fundamental group is F_n.

3. In $\mathbf{R}^3$ with Euclidean coordinates, consider the differential 1-form $\alpha = \cos y\, dz - \sin y\, dx$ and the vector field $v = \cos y \frac{\partial}{\partial z} - \sin y \frac{\partial}{\partial x}$. Prove that α is invariant under the flow of v.

4. Let M be a compact n-manifold without boundary. Let $f : M \to \mathbf{R}^n$ be a smooth map. Show that for every point $x \in M$ outside a measure zero subset, the preimage $f^{-1}(x)$ consists of an even number of points. Hint: If you can replace $\mathbf{R}^n$ with a compact manifold, you can use degree theory.

5. Suppose that M is an orientable n-manifold and ω is an $n-1$-form on M. If the integral of ω over every $(n-1)$-dimensional submanifold of M is zero, show that $d\omega = 0$.

6. Recall that the cone over a space X is the quotient $CX = (X \times [0,1])/(X \times \{0\})$. If the cone over S^n is homeomorphic to the cone over X, show that X is either contractible or homotopy equivalent to S^n.

7. Let X be a CW complex obtained by attaching a 2-cell to $S^1 \times \mathbf{R}P^2$. Show that $\pi_1(X)$ is not a finite group of odd order.

8. Let M be the open Möbius band, that is, the total space of a nontrivial real line bundle over the circle. Show that $M \times M$ is not diffeomorphic to $S^1 \times S^1 \times \mathbb{R}^2$.