Probability Comprehensive Exam
Spring 2026

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\Pi \subseteq \mathcal{F}$ be a π-system. Suppose that $A \in \mathcal{F}$ is independent of Π in the sense that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $B \in \Pi$. Show that, if $A \in \sigma(\Pi)$, then $\mathbb{P}(A)$ is either zero or one.

2. If $a \in \mathbf{R}$ and $b > 0$, we write $\mathcal{L}_{a,b}$ for the lattice $\mathcal{L}_{a,b} = \{\dots, a-2b, a-b, a, a+b, a+2b, \dots\}$. We say that a random variable X is *distributed on a lattice* if there exist a, b such that $\mathbb{P}(X \in \mathcal{L}_{a,b}) = 1$. Show that X is distributed on a lattice if and only if its characteristic function φ_X satisfies $|\varphi_X(t_*)| = 1$ for some $t_* \neq 0$.

3. Let X and Y be independent random variables defined on the same probability space. Show that, if X and Y are independent and nonnegative, then we have a backwards triangle inequality in the sense that $\|X + Y\|_\infty \geq \|X\|_\infty + \|Y\|_\infty$.

4. Let X be a nonnegative random variable with $\mathrm{Var}(X) \leq 1/2$. Show that

$$\mathbb{P}(-1 + \mathbb{E}[X] \leq X \leq 2\mathbb{E}[X]) \geq 1/2.$$

   You may wish to consider casework based on the value of $\mathbb{E}[X]$.

5. Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume further that X_1 has a density f which is continuous, bounded, even and such that $f(0) > 0$.

   (i) For each $n \geq 1$, let

$$Y_n = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{X_k}.$$

   Show that the sequence $(Y_n)_{n\geq 1}$ converges in law. Can you identify the limiting law?

   (ii) Show that the inverse of a standard Cauchy random variable has also a standard Cauchy law, and then obtain the limiting law for convergence in distribution of the empirical harmonic mean

$$H_n = \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{X_k} \right)^{-1}.$$

   Hint: Recall that a (standard) Cauchy random variable has density given by $\frac{1}{\pi(1+x^2)}$, $x \in \mathbf{R}$. You may also use that $\int_0^\infty \frac{\cos u - 1}{u^2}\,\mathrm{d}u = -\frac{\pi}{2}$.

6. Let $(X_n)_{n\geq 1}$ be a sequence of identically distributed (not necessarily independent) random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, all taking values in $\{0, 1, 2, 3, \dots\}$. Let $R_n = |\{X_1, X_2, \dots, X_n\}|$ be the cardinality of the (random) set $\{X_1, X_2, \dots, X_n\}$.

   (i) Show that for any $N \in \{0, 1, 2, \dots\}$, $\mathbb{E}R_n \leq N + n\mathbb{P}(X_1 \geq N)$, and deduce the existence and value of $\lim_{n\to\infty} \mathbb{E}R_n/n$.

   (ii) Now assume that X_1 has finite expectation. Show that $\lim_{n\to+\infty} \mathbb{E}R_n/\sqrt{n} = 0$.

7. (i) Let Z be a standard normal random variable. Show that, for all $x > 0$, we have $\mathbb{P}(Z \geq x) \leq 1/x$.

   (ii) Recall that the Kolmogorov distance d_K between the probability laws μ_X and μ_Y of the random variables X and Y is given by

$$d_K(\mu_X, \mu_Y) = \sup_{x\in\mathbf{R}} |F_X(x) - F_Y(x)|,$$

   where $F_X(x) = \mu_X((-\infty, x])$ and $F_Y(x) = \mu_Y((-\infty, x])$. Let now γ_t be a centered Gaussian measure of variance t^2, and let μ be absolutely continuous with density bounded by C. Show that

$$d_K(\gamma_t * \mu, \mu) \leq 2\sqrt{Ct}.$$

   (**Hint:** Start by using part (i) to show that, for any $x \in \mathbf{R}$ and $h > 0$, $F_{\gamma_t*\mu}(x) - F_\mu(x) \leq Ch + t/h$.)

8. Let $X_1, X_2, \dots,$ be independent random variables where each X_n, $n \geq 1$, has law

$$\mathbb{P}_{X_n} = \frac{1}{2}\left(1 - \frac{1}{2^n}\right)(\delta_{-1} + \delta_1) + \frac{1}{2^{n+1}}(\delta_{-2^n} + \delta_{2^n}).$$

   (i) Does $\sum_{n=1}^{+\infty} \frac{\operatorname{Var} X_n}{n^2}$ converge?

   (ii) Does $(X_n)_{n\geq 1}$ satisfy the strong law of large numbers?