Discrete Mathematics Comprehensive Exam
Spring 2026

Student Number: ☐

Instructions: Complete **exactly 5** of the given 6 problems and **circle** their numbers below. The uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

All graphs in the following problems are finite and simple, except where specified.

1. Let G be triangle-free graph. Prove that $\chi(G) \leqslant 2\sqrt{|V(G)|}$.

2. Suppose G is a 3-connected nonplanar graph and $e = uv \in E(G)$ such that G/e (the graph obtained from G by contracting e) is 3-connected and planar. Show that G contains a subdivision of K_5 or $K_{3,3}$ in which both u and v are branch vertices.

3. Let G be a graph on n vertices and assume that $K_4^- \not\subseteq G$, where K_4^- is the graph obatined from K_4 by removing an edge. Show that $|E(G)| \leqslant n^2/4$ for sufficiently large n.

4. If the edges of a complete graph are colored by 3 colors, we say a vertex subset U is color-avoiding if at most two of the colors appear on edges inside U. Prove that there exists $c > 0$ such that for all sufficiently large n, there is an edge coloring of the complete graph on $c \cdot (\frac{3}{2})^{n/2}$ vertices with no color-avoiding subset of size n.

5. An induced cherry in a graph G is a triple of vertices u, v, w such that $u \sim v$, $v \sim w$, and $u \not\sim w$. Find (with proof) a function $f : \mathbb{N} \to \{0, 1\}$ such that for $p \in [0, 1/2]$:
    1. If $p = o(f(n))$ then w.h.p. $G(n, p)$ does not contain an induced cherry.
    2. If $p = \omega(f(n))$ then w.h.p. $G(n, p)$ contains an induced cherry.

6. Let $n \geqslant 1$ and let $x_1, \ldots, x_n$ be nonnegative integers less than t, where $t = \frac{2^{n/2}}{100n^{1/4}}$. Prove that there exists two distinct, disjoint subsets $I, J \subseteq [n]$ for which
$$\sum_{i \in I} x_i^2 = \sum_{j \in J} x_j^2.$$