Differential Equations Comprehensive Exam
Spring 2026

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Consider
$$x' = A(t)x + g(t), \qquad \text{where } A(t) = \begin{pmatrix} 1+\cos t & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix}$$
and $g \in C^0$ is 2π-periodic. Prove that there exists a unique 2π-periodic solution.

2. Consider
$$x' = A(t)x, \qquad \text{where } A(t) = \begin{pmatrix} -1-\sin t & a(t)+\cos^2 t \\ b(t)+\sin t & 3+\sin t \end{pmatrix},$$
and $a, b \in C^0(R, R)$. Prove the system is Lyapunov unstable.

3. Consider
$$x'' + (2+\sin(\pi x))x' - x + x^2 = 0, \qquad x \in R.$$
   (a) Find all equilibria and analyze their stable, asymptotic stability, exponential stability, instability, etc.
   (b) Prove there exists a heteroclinic orbit, i.e. an orbit which converges to different equilibria as $t \to \pm\infty$.

4. Consider
$$\begin{cases} x' = \epsilon x + x^2 y + y - (x^4+y^4)x \\ y' = \epsilon y - x^3 - x - (x^4+y^4)y. \end{cases}$$
where $\epsilon > 0$.
   (a) For $\epsilon = 0$, prove all solutions converge to 0 as $t \to +\infty$.
   (b) For $\epsilon > 0$, prove there exists a limit cycle.

5. Consider the following initial value problem
$$\begin{cases} u_t + uu_x = -2u, \ x \in \mathbf{R}, t > 0, \\ u(x,0) = \dfrac{A}{1+x^2}. \end{cases}$$
where $A \geq 0$ is a constant. Determine the optimal range of A so that this initial value problem admits a unique global (that is for all $x \in \mathbf{R}$ and $t \geq 0$) C^1 solution $u(x,t)$.

6. Let B be a ball in $\mathbf{R}^n$ ($n \geq 2$) with radius R with boundary ∂B. $f(x)$ and $g(x)$ are continuous functions such that

$$\max\{\|f(x)\|_{L^\infty(B)}, \|g(x)\|_{L^\infty(\partial B)}\} = M$$

for some positive constant M. If u is a smooth solution of the following problem

$$\begin{cases} -\Delta u = f, \ x \in B, \\ u = g, on\ \partial B. \end{cases}$$

Prove that there exists a constant C, depending only on n and R, such that

$$\max_{\bar{B}} |u| \leq CM$$

7. Let $f(x,t)$, $g(x)$, and $h(x)$ be bounded and smooth functions of their arguments. $g(0) = g(1) = h(0) = h(1) = 0$. For any finite constant $c \in \mathbf{R}$ and any non-negative constant $d \in \mathbf{R}$, prove that there is at most one solution $u \in C^2([0,1] \times [0,\infty))$ to the following problem

$$\begin{cases} u_{tt} - u_{xx} + cu_t + du = f(x,t), \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0 \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ for\ x \in (0,1), \end{cases}$$

8. Assume that $u \in C^{2,1}((0,\pi) \times (0,\infty))$ solves

$$\begin{cases} u_t - u_{xx} = au, \ x \in (0,\pi), \ t > 0, \\ u(0,t) = u(\pi,t) = 0, \ t > 0, \\ u(x,0) = f(x), \ for\ x \in (0,\pi), \end{cases}$$

where $a < 1$ is a constant, and $f(x) \in C_0^\infty(0,\pi)$, that is $f(x)$ has compact support in $(0,\pi)$. Prove that

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{L^2([0,\pi])} = 0.$$

(Hint: It is possible to use the separation of variables method.)