Analysis Comprehensive Exam
Spring 2026

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTE:

- All scalars in this exam are real unless explicitly stated otherwise.
- All functions in this exam are (extended) real-valued unless explicitly stated otherwise.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^n$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- For $1 \le p < \infty$, and a measurable function f on a measurable set E, set $\|f\|_p^p = \displaystyle\int_E |f(x)|^p \, dx$.

1. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of nonnegative Lebesgue measurable functions such that

$$\int_0^1 f_j(x)\,dx \;\le\; \frac{1}{j^2}, \qquad j = 1, 2, 3, \ldots$$

Define

$$g_n(x) := \prod_{j=1}^{n} f_j(x), \qquad x \in [0,1].$$

Prove that $g_n(x) \to 0$ for almost every $x \in [0,1]$.

2. Let $f \in L^1([0,1])$, and define

$$F(x) := \int_0^x f(t)\,dt, \qquad x \in [0,1].$$

Prove that there exists a sequence of continuously differentiable functions F_n on $[0,1]$ such that

$$\sup_{x\in[0,1]} |F_n(x) - F(x)| \longrightarrow 0 \quad \text{as } n \to \infty.$$

3. Let $1 \le p < \infty$. For each $n \in \mathbb{N}$, define a linear operator

$$T_n : L^p(0,1) \longrightarrow L^p(0,1)$$

by

$$T_n f(x) \;=\; n \int_x^{x+\frac{1}{n}} f(t)\,dt,$$

where f is extended by zero outside $(0,1)$.

Prove that

$$\sup_{n\in\mathbb{N}} \|T_n\|_{L^p \to L^p} = 1.$$

4. Let $f_n \in L^3([0,1])$ satisfy:

- $f_n(x) \to f(x)$ almost everywhere on $[0,1]$,
- $\sup_n \|f_n\|_{L^3([0,1])} \le 1$.

(a) Show that for every measurable set $E \subset [0,1]$ and every n,

$$\int_E |f_n(x) - f(x)|^2\,dx \;\le\; 4\,|E|^{1/3}.$$

(b) Deduce that

$$\|f_n - f\|_{L^2([0,1])} \;\longrightarrow\; 0.$$

5. Let $\phi : \mathbb{Z} \to (0, \infty)$, that is, ϕ is a positive function defined on the integers. Assume also that

$$\sum_{k=1}^{\infty} k^2 \phi(k)^2 < \infty.$$

Let $\mathcal{A} \subset \mathbb{R}^2$ be the set of all $(x, y) \in \mathbb{R}^2$ such that for infinitely many $k \geq 1$, there exist a pair of rational numbers $\left(\frac{j}{k}, \frac{\ell}{k}\right)$ with

$$\left| (x, y) - \left(\frac{j}{k}, \frac{\ell}{k} \right) \right| < \phi(k). \tag{1}$$

Show that $|\mathcal{A}| = 0$.

6. Let E be a set in $\mathbb{R}^n$ with $0 < |E|_e < \infty$, where the subscript e denotes exterior (or outer) Lebesgue measure. Let $0 < \theta < 1$. Show that there is a set $E_\theta \subset E$ with

$$|E_\theta|_e = \theta |E|_e.$$

7. Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with $|E| < \infty$. Let $f_k : E \to \mathbb{R}$ be measurable for $k \geq 1$. Assume for each $\mathbf{x} \in E$, there exists $M_{\mathbf{x}} < \infty$ such that

$$\sup_{k \geq 1} |f_k(\mathbf{x})| \exp\left(-|\mathbf{x}|^2\right) \leq M_{\mathbf{x}}.$$

Let $\varepsilon > 0$. Show there is a finite number M and a closed set $F \subset E$ with $|E \backslash F| < \varepsilon$ and

$$|f_k(\mathbf{x})| \exp\left(-|\mathbf{x}|^2\right) \leq M \text{ for all } k \geq 1 \text{ and } \mathbf{x} \in F.$$

8. Let (S, Σ, ν) be a measure space. Assume that

$$S = \bigcup_{n=1}^{\infty} E_n$$

where the $\{E_n\}_{n=1}^{\infty}$ are disjoint measurable sets, each with $\nu(E_n) < \infty$. Define μ on S by

$$\mu(B) = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \frac{\nu(B \cap E_n)}{\nu(E_n)}, B \in \Sigma.$$

(a) Show μ is a finite measure on S.
(b) Show μ is absolutely continuous on S w.r.t. ν and ν is absolutely continuous on S w.r.t. μ.
(c) Find a function $f : S \to \mathbb{R}$ such that for all $A \in \Sigma$,

$$\mu(A) = \int_A f \, d\nu.$$