Algebra Comprehensive Exam
Spring 2026

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. How many groups of order $1225 = 25 \cdot 49$ are there up to isomorphism? [**Hint:** First show that any such group must be abelian.]

2. Let G be the group of all invertible upper-triangular 2×2 real matrices (with group law matrix multiplication). Let H be the subset of G consisting of all elements of the form g^2 with $g \in G$. Show that H is a subgroup of G and compute its index, i.e., the number of (left) cosets of H in G.

3. Let $m \geq 2$ be an integer. Show that the polynomial
$$f(X,Y) = X^m + Y^m + 1$$
is irreducible in $\mathbb{C}[X,Y]$. [**Hint:** Eisenstein's criterion holds in any UFD, for example $\mathbb{C}[Y]$.]

4. Two polynomials $f, g \in R[t]$ with coefficients in a commutative ring R with identity are called *relatively prime over* R if the ideal of $R[t]$ generated by f and g contains 1. Suppose that $f, g \in \mathbf{Z}[t]$ are non-constant monic polynomials which are relatively prime over $\mathbf{Q}$, and that the reductions of f and g modulo p are relatively prime over $\mathbf{Z}/p\mathbf{Z}$ for all prime numbers p. Prove that f and g are relatively prime over $\mathbf{Z}$.

5. Let $L/\mathbb{Q}$ be an algebraic extension. Suppose that for all $a \in L$, the extension $\mathbb{Q}(a)/\mathbb{Q}$ has degree at most 2. Show that $[L : \mathbb{Q}] \leq 2$.

6. Let F be a finite field with cardinality q. For how many $a \in F$ does the polynomial $x^5 - a$ have a root in F? Express your answer in terms of q.

7.
   1. Suppose n is a positive integer and that A is an $n \times n$ matrix with real entries such that $A^2 = -I$. Prove that n is even and that $\det(A) = 1$.
   2. Show that for every even positive integer n, there exists an $n \times n$ square matrix A with real entries such that $A^2 = -I$, where I denotes the $n \times n$ identity matrix.

8. Let $R = \mathbb{Q}[x]/\langle x^5 \rangle$.
   1. Show that R is a free $\mathbb{Q}$-module of rank 5
   2. Show that R is a torsion module as a $\mathbb{Q}[x]$-module.