Topology Comprehensive Exam
Fall 2026

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. The graph of a function $f : M \to \mathbf{R}$ is $\Gamma_f = \{(x, z) \in M \times \mathbf{R} : z = f(x)\}$. Given two smooth functions $f, g : M \to \mathbf{R}$ give necessary and sufficient conditions for Γ_f and Γ_g to be transverse in $M \times \mathbf{R}$.

2. Let A be a subset of S^2 that consists of n points. Show that the quotient space S^2/A is homotopy equivalent to a wedge of spheres, and determine the spheres in the wedge. If your argument is a series of pictures, add some comments explaining what you are doing.

3. Let η be a nowhere zero 1-form on a manifold M such that $\eta \wedge d\eta = 0$. Let P be the subbundle of TM whose fiber over $x \in M$ is the kernel of η at x. Show that if two vector fields v, w are sections of P then $[v, w]$ is also a section of P.

4. Let M be a compact connected oriented $(n-1)$-manifold without boundary, where $n > 1$, and $f : M \to \mathbf{R}^n$ be a smooth map. Let α be an $(n-1)$-form on $\mathbf{R}^n$ such that $d\alpha$ is nowhere zero. If $\int_M f^*\alpha = 0$, show that f is not an embedding. Hint: What can you say about the components of $\mathbf{R}^n - M$.

5. Let K be the Klein bottle thought of as a square with the opposite sides identified, one with a twist and the other one without. Let γ be a loop in K corresponding to the side of the square that is identified without a twist. Show that the fundamental group of the space obtained from K by attaching a 2-disk along the loop $\gamma \cdot \gamma$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2$. As usual $\gamma \cdot \gamma$ is the product of γ and γ.

6. Find all $k \in \{0, 1, 2, 3, 4\}$ such that for every proper smooth k-dimensional submanifold M of $\mathbf{R}^5$ each continuous map from S^1 to $\mathbf{R}^n \setminus M$ is null-homotopic. Here *proper* means that the intersection of M with any compact set of $\mathbf{R}^5$ is compact in M.

7. Let v be a vector field on a manifold M such that v is non-zero at a point $p \in M$. Show that there is a coordinate chart around p with coordinates $(x_1, \ldots, x_n)$ such that in these coordinates v is $\frac{\partial}{\partial x_1}$. Hint: Start from any coordinates about p and use the flow of v to generate a new coordinate chart.

8. Show that every map $f : \mathbb{R}P^2 \to \mathbb{R}P^3$ that induces a trivial homomorphism of fundamental groups is null-homotopic. Hint: Sard.