Probability Comprehensive Exam
Fall 2026

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $X_1, X_2, \dots$ be i.i.d. random variables defined on the same underlying probability space, and let $S_n = X_1 + \cdots + X_n$. Suppose that

$$\mathbb{P}\left(\limsup_{n\to\infty}\left|\frac{S_n}{n}\right| < +\infty\right) > 0.$$

Show that $\mathbb{E}[|X_1|] < +\infty$.

2. Let $X_1, X_2, \dots$ be i.i.d. random variables on the same underlying probability space, each with characteristic function φ, and let $S_n = X_1 + \cdots + X_n$. Show that, if φ is differentiable at zero with $\varphi'(0) = z \in \mathbf{C}$, then necessarily z is purely imaginary, $z = \mathrm{i}a$ for some $a \in \mathbf{R}$, and $\frac{S_n}{n}$ converges in probability to a as $n \to \infty$. Notice we are *not* assuming that the X_i's are integrable.

3. Let X and Y be independent random variables defined on the same underlying probability space. Show that, if $\mathbb{E}[|X|] = +\infty$, then $\mathbb{E}[|X - Y|] = +\infty$.

4. Let $X_1, X_2, \dots$ be i.i.d. random variables on the same underlying probability space, centered and with unit variance, and let $S_n = X_1 + \cdots + X_n$. Show that

$$\mathbb{P}\left(\limsup_{n\to\infty}\frac{S_n}{\sqrt{n}} = +\infty\right) = 1.$$

5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with the following property: for any $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, there exists $B \in \mathcal{F}$ with $B \subset A$ and $0 < \mathbb{P}(B) < \mathbb{P}(A)$. Let $A \in \mathcal{F}$ have $\mathbb{P}(A) > 0$ and let $n \geq 1$. Prove that there exists $B \in \mathcal{F}$ with $B \subset A$ and $0 < \mathbb{P}(B) \leq 1/n$.

6. Let (X_n) be a sequence of random variables that converge in distribution to a random variable X. Suppose that there are real constants $(c_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty}\mathbb{P}(X_n = c_n) = 1.$$

Show that the sequence $(c_n)_{n=1}^{\infty}$ converges to some $c \in \mathbf{R}$, and that $\mathbb{P}(X = c) = 1$.

7. (a) Let X be a random variable and suppose that for some $a, b, c \in \mathbb{R}$ with $a \leq b$ and $c > 0$, we have

$$\mathbb{P}(X \leq a) \geq c \text{ and } \mathbb{P}(X \geq b) \geq c.$$

Show that $\operatorname{Var} X \geq c(b-a)^2/4$.

(b) Use part 1 to show that if $X_1, \ldots, X_n$ are i.i.d. random variables uniformly distributed on $(0,1)$, then there exists $c' > 0$ such that for all large n, we have

$$\operatorname{Var}(\max\{X_1, \ldots, X_n\}) \geq \frac{c'}{n^2}.$$

8. Let $(X_n)_{n=1}^{\infty}$ be an independent sequence of random variables that are each unbounded above and let $M \in \mathbb{R}$. Prove that the following are equivalent.

(a) $\sum_{n=1}^{\infty} \mathbb{P}(X_n \leq M) = \infty$.

(b) $\mathbb{P}\left(X_n \leq M \text{ for some } n \geq 1\right) = 1$.