Analysis Comprehensive Exam
Fall 2026

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTE:

- All scalars in this exam are real unless explicitly stated otherwise.
- All functions in this exam are (extended) real-valued unless explicitly stated otherwise.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set A is denoted by χ_A.
- The L^1-norm of a measurable function f on a measurable set E is $\|f\|_1 = \displaystyle\int_E |f(x)|\,dx$.

1. Apply Fubini's Theorem to
$$f(x,y) = e^{-x}\cos(2xy)$$
on $E = (0,\infty)\times(0,2)$ to prove
$$\int_0^\infty e^{-x}\frac{\sin 4x}{x}dx = \frac{1}{2}\tan^{-1}(4).$$
Here $\tan^{-1}$ is the inverse function of $\tan$.

2. Suppose that $f \in L^1(\mathbf{R})$ is absolutely continuous on every finite interval $[a,b]$. Prove that if
$$\lim_{t\to 0^+}\int_{-\infty}^{\infty}\left|\frac{f(x+t)-f(x)}{t}\right| dx = 0,$$
then $f = 0$ a.e.

3. For $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be integrable. Assume that
$$\lim_{k\to\infty} f_k = f \text{ a.e. in } [0,1],$$
Assume also that $\forall\varepsilon > 0$, there exists $\delta > 0$ such that
$$E \subset [0,1] \text{ and } |E| < \delta \Rightarrow \int_E f_k^2 < \varepsilon \text{ for all } k \geq 1.$$
Prove that
$$\lim_{k\to\infty}\int_0^1 |f_k - f|^2 = 0. \tag{1}$$

4. (a) Use the Weierstrass Approximation Theorem to prove that the set of polynomials is dense in $L^2[a,b]$.

   (b) Does there exist a function $f \in L^2[a,b]$ such that
$$\int_a^b xf(x)\,dx = 1 \qquad \text{and} \qquad \int_a^b x^k f(x)\,dx = 0, \quad \text{for } k = 0,2,3,\ldots?$$

5. Let X be a Banach space, Y be a normed linear space, and $B : X\times Y \mapsto \mathbb{R}$ be a bilinear function (that is, it is linear in each of its two variables). Suppose that for each $x \in X$ there exists a constant $C(x) \geq 0$ such that
$$|B(x,y)| \leq C(x)\,\|y\| \text{ for all } y \in Y$$
and for each $y \in Y$, there exists $C(y) \geq 0$ such that
$$|B(x,y)| \leq C(y)\,\|x\| \text{ for all } x \in X.$$
Show that then there exists a constant $C \geq 0$ such that
$$|B(x,y)| \leq C\|x\|\|y\|$$
for all $x \in X$ and all $y \in Y$.

6. Let H be the set of all absolutely continuous functions $f \in \mathrm{AC}[a,b]$ such that $f(a) = 0$ and $f' \in L^2[a,b]$. Prove that H is a Hilbert space with respect to the inner product

$$\langle f, g \rangle \;=\; \int_a^b f'(x)\,\overline{g'(x)}\,dx.$$

7. Let $n \geq 1$. Let μ be a positive measure on $\mathbb{R}$ with infinitely many points in its support. Assume that every polynomial of degree $\leq n$ is $\mu-$integrable over $\mathbb{R}$. Let P be a polynomial of degree n such that for $j = 0, 1, 2, ..., n-1$,

$$\int P(x)\, x^j d\mu(x) = 0.$$

(a) Prove that P has n distinct zeros in $\mathbb{R}$.

(b) Prove that if $[a,b]$ is an interval that lies outside the support of μ (that is $\mu([a,b]) = 0$), then P can have at most one zero in $[a,b]$.

8. Prove that if $f \in L^1(\mathbf{R}) \cap L^4(\mathbf{R})$, then for every $1 \leq p \leq 4$ we have both

(a) $f \in L^p(\mathbf{R})$, and

(b) $\lim_{k\to\infty} k^p \left|\{x \in \mathbf{R} : |f(x)| > k\}\right| \;=\; 0.$