Algebra Comprehensive Exam
Fall 2026

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a simple group of order $|G| = 60 = 2^2 \cdot 3 \cdot 5$. Prove that there are exactly 10 3-Sylow subgroups of G.

2. Let K be a field and $A := K[x^2, xy, y^2] \subset K[x, y]$.
   (a) Is the ring A a unique factorization domain?
   (b) Is it a principal ideal domain?

3. Let G be an abelian group. Note that we do not require G to be finitely generated. Show that $G/3G$ is an $\mathbf{F}_3$ vector space, where $\mathbf{F}_3$ is the field with three elements.

4. (a) Let I be an ideal in a principal ideal domain R such that $I^2 = I$. Show that $I = \langle 0 \rangle$ or $I = R$.
   (b) Give an example of an ideal I in a unital commutative ring R such that $I^2 = I$ but I is not (0) or R. Justify your answer.

5. Let A be a 5×5 matrix with complex entries and characteristic polynomial $x^2(x-1)(x+1)^2$. Determine the possible characteristic and minimal polynomials of A^2.

6. Let G be a finite group. Fix a prime number p. Show that the following are equivalent:
   (a) The order of G is not a power of p.
   (b) There is a finite set S of size at least 2 and relatively prime to p, such that G acts transitively on S (i.e. there is only one orbit in S under the action of G).

7. Let K be the splitting field over $(x^3 - 2)(x^2 - 5)$ over $\mathbf{Q}$. Compute $[K : \mathbf{Q}]$.

8. (a) Compute $[\mathbf{F}_{64} : \mathbf{F}_2]$. Write $\prod_{a \in \mathbf{F}_{64}} (x - a)$ in simplified form. [Note that $64 = 2^6$.]
   (b) How many irreducible polynomials of degree 6 are in $\mathbf{F}_2[x]$?