Probability Comprehensive Exam
Spring 2025

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $A_1, A_2, \dots$ be independent events with $\mathbb{P}(A_n) = 1/n$ for all $n \geq 1$. Define a sequence of random variables $(N_n)_{n\geq 1}$ by

$$N_n(\omega) = \#\{k = 2^n, ..., 2^{n+1} - 1 : \omega \in A_k\}.$$

Prove that

$$\liminf_{n\to\infty} \mathbb{P}(N_n \geq 1000) > 0.$$

2. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(A_n)_{n\geq 1}$ be a sequence of events such that $\sum_{n=1}^{+\infty} \mathbb{P}(A_n) = +\infty$ and

$$\liminf_{n\to+\infty} \left(\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{P}(A_i \cap A_j)}{(\sum_{i=1}^{n} P(A_i))^2} \right) = c < +\infty.$$

Show that $\mathbb{P}(A_n \text{ i.o.}) \geq 1/c > 0$.

Hint : One way to prove the result is to use the Paley-Zygmund inequality which asserts that if $X \geq 0$ is a non-degenerate random variable with $0 < \mathbb{E}(X^2) < +\infty$ and if $0 < \lambda < 1$, then $\mathbb{P}(X \geq \lambda \mathbb{E}X) \geq (1-\lambda)^2 (\mathbb{E}X)^2 / \mathbb{E}(X^2)$.

3. Let X and Y be random variables such that $\mathbb{E}|X| < \infty$ and $\mathbb{E}|Y| < \infty$. Prove that

$$\int_{-\infty}^{\infty} [\mathbb{P}(X < x \leq Y) - \mathbb{P}(Y < x \leq X)] \, \mathrm{d}x = \mathbb{E}Y - \mathbb{E}X.$$

4. Let $(X_n)_{n\geq 1}$ be a sequence of Gaussian random variables that converge in L^2 to some random variable X. Prove that X is either (a) almost surely equal to a constant or (b) a Gaussian random variable.

5. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. standard Gaussian random variables and define

$$T = \min\{n \geq 1 : X_1 + \cdots + X_n < n \log n\}.$$

Prove that $\mathbb{E}e^{\alpha T} < \infty$ for every $\alpha > 0$.

6. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with finite first moment and such that $\mathbb{E}X_n = 0$, for all $n \geq 1$. Show that $\sum_{n=1}^{+\infty} X_n^2$ converges almost surely if and only if

$$\sum_{n=1}^{+\infty} \mathbb{E}\left(\frac{X_n^2}{1 + X_n^2}\right) < +\infty.$$

7. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables such that $\mathbb{P}(X_n = 1) = 1/n$ and $\mathbb{P}(X_n = 0) = 1 - 1/n$. Show that properly centered and normalized, $S_n = \sum_{k=1}^{n} X_k$ converges in distribution towards a non-degenerate random variable that you will identify.

8. Let $(X_n)_{n\geq 1}$ be a sequence of iid random variables with finite second moment. Let $\mathbb{E}X_1 = 0$, let $\mathbb{E}X_1^2 = \sigma^2$, $0 < \sigma^2 < +\infty$ and let $S_n = \sum_{k=1}^{n} X_k$. Show that

$$\lim_{n\to+\infty} \frac{\mathbb{E}|S_n|}{\sqrt{n}} = 2 \lim_{n\to+\infty} \frac{\mathbb{E}S_n^+}{\sqrt{n}} = \sqrt{\frac{2\sigma^2}{\pi}},$$

where as usual $S_n^+ = \max(S_n, 0)$.