Differential Equations Comprehensive Exam
Spring 2025

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Consider the initial value problem

$$\begin{cases} \dot{x} = x^2 \sin(t) \\ x(0) = x_0 \end{cases}$$

on $\mathbf{R}$ and let $x(t) : [0, T(x_0)) \to \mathbf{R}$ denote its maximal solution in forward-time, $T(x_0) \in (0, \infty]$. Find $T(x_0)$ for all $x_0 \in \mathbf{R}$.

2. Consider the ODE

$$\begin{cases} \dot{x} = ax - y + xy^2 \\ \dot{y} = x + ay + y^3 \end{cases} .$$

Show that if $a > 0$, then there are no nontrivial periodic orbits.

Hint: Let $r = \sqrt{x^2 + y^2}$ and estimate $\frac{d}{dt}(\log r)$ from below.

3. Consider a planar ODE $\dot{x} = F(x)$ with smooth vector field $F : \mathbf{R}^2 \to \mathbf{R}$. Assume $F(0) = 0$, and suppose V is a Lyapunov function for F at 0, i.e., there is a smooth function $V : \mathbf{R}^d \to [0, \infty)$ and a radius $r > 0$ such that

   (a) $V > 0$ on $U \setminus \{0\}$; and

   (b) $\dot{V} \leq 0$ on U.

   Here $U = B_r(0)$ is the ball of radius r centered at 0.

   Assume $\{x \in U : \dot{V} = 0\} \subset \Sigma$, where Σ is a closed, smooth curve containing the origin, and assume moreover that $\Sigma \setminus \{0\}$ is transversal to F. Prove that 0 is asymptotically stable.

4. Consider the 2π-periodic linear ODE

$$\dot{x} = A(t)x = \begin{pmatrix} \sin^2 t + 1 & e^{\sin t} \\ e^{\cos t} & \cos^2 t \end{pmatrix} x .$$

Show that this ODE admits a solution $x(t)$ for which

$$\lim_{t \to \infty} \frac{1}{t} \log |x(t)| \geq 1 .$$

5. Fix a bounded subset Ω of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, and for $T \geq 1$ write $\Omega_T = \Omega \times (0, T]$ with boundary $\Gamma_T = (\Omega \times \{t = 0\}) \cup (\partial\Omega \times [0, T])$. Assume $u(x, t)$ has continuous 2nd order derivatives in $x \in \bar{\Omega}$ and continuous first order derivatives in $t \in [0, T]$, and that $u(x, t)$ is the solution of

$$\begin{cases} u_t - \Delta u = f(x, t), \ (x, t) \in \Omega_T , \\ u(x, t)|_{\Gamma_T} = \phi(x, t) , \end{cases}$$

where f and ϕ are bounded functions on $\bar{\Omega}_T$. Define

$$F = \sup_{\bar{\Omega}_T} |f|, \qquad B = \sup_{\Gamma_T} |\phi|.$$

Prove that

$$\max_{\bar{\Omega}_T} u \leq FT + B.$$

6. Assume $u \in C^2([0,1] \times [0,\infty))$ is a classical solution to the following problem:

$$\begin{cases} u_{tt} - u_{xx} + u_t = -\frac{1}{4}u, \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0 \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ for \ x \in (0,1), \end{cases},$$

where g, h are C^2 functions such that

$$g(0) = g(1) = h(0) = h(1) = 0\,.$$

Prove that there exists a constant C such that

$$\|u(\cdot,t)\|_{L^2_x([0,1])} \leq Ce^{-\frac{1}{2}t}.$$

7. Identify the types of the following equation

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0,$$

and then transfer it into standard form.

8. For the following problem

$$\begin{cases} u_t + u^3 u_x = 0, \ x \in \mathbf{R}, t > 0, \\ u(x,0) = \dfrac{1}{1+x^2}, \end{cases}$$

Find the time when the C^1 solution $u(x,t)$ blows up first.

Hint: Look for a blowup in $\|u_x(x,t)\|_{L^\infty}$.