Numerical Analysis Comprehensive Exam
Spring 2025

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Sarah considers to compute $\sqrt{5}$ by iterative procedures, assuming she starts the iterations with a value close to $\sqrt{5}$.

   (i) If she uses $x_{n+1} = x_n^2 + x_n - 5$ to compute, will she be successful in getting the value?

   (ii) If she wants to use the Newton's method, can you give her the iteration formula? What is the rate of convergence? If she starts with the initial guess as $x_0 = 2.1$, how many iterations will it take to obtain an approximation with error smaller than 10^{-12}?

   You must justify your answer.

2. Let $f(x, y)$ be a sufficiently smooth function. Suppose the values of f are only known at $(0, 0)$, $(0, h)$, (h, h) and $(h, 0)$ as A, B, C and D respectively, $h > 0$.
   (a) Find a numerical approximation of $f(x, y)$, $0 \leq x, y \leq 1$, so that the approximation error is $O(h^2)$ as $h \to 0$ (known as the second order approximation).
   (b) Find a second order numerical approximation of $\frac{\partial f}{\partial x}(h/2, h/2)$, and justify the order of the approximation error.

3. Consider solving an ODE: $y_t = f(t, y)$. Given a 2nd order scheme $y^{n+1} = L(y^n, \Delta t)$ for solving the ODE for a time step $\Delta t = t_{n+1} - t_n$, design a new scheme as

$$y^{n+1} = y^n + (t_{n+1} - t_n)\frac{1}{6}\{f(t_n, y^n) + 4f(t_{n+1/2}, \tilde{y}^{n+1/2}) + f(t_{n+1}, \tilde{y}^{n+1})\},$$

where $\tilde{y}^{n+1/2}$ and $\tilde{y}^{n+1}$ are computed by scheme L with initial value y^n for time step sizes $\frac{1}{2}\Delta t$ and Δt respectively. Show that the new scheme is 3rd order accurate.

4. Suppose a scheme can be written as

$$\sum_{j=-m_1}^{m_1} \alpha_j U_{i+j}^{n+1} = \sum_{j=-m_2}^{m_2} \beta_j U_{i+j}^{n}$$

where α_0 and at least two other coefficients are non zero. Find conditions for the coefficients so that the scheme is stable in l^∞ and prove your claim.

5. Consider the 1D Poisson equation $-u''(x) = f(x)$, $x \in (0, 1)$, with Dirichlet boundary conditions $u(0) = a$ and $u(1) = b$. Given a partition $0 = x_0 < x_1 < \cdots < x_n = b$, define the trial and test spaces and the piecewise linear continuous finite element method for solving the equation. Write down the formula for the resulted linear system and show that it has a unique solution.

6. Given $x \in \mathbb{C}^2$, determine a unitary matrix of the form

$$Q = \begin{bmatrix} c & \bar{s} \\ -s & c \end{bmatrix},$$

where $c \in \mathbb{R}$ and $c^2 + |s|^2 = 1$, such that the second entry of Q^*x is zero. Can you design an algorithm to use the obtained formula for Q to compute the QR factorization of a $n \times n$ upper Hessenberg matrix H (an upper Hessenberg matrix has zero entries below the first subdiagonal)? If preferred, you may write your algorithm in a psuedo-code fashion. What is the leading order for the computational cost of your algorithm.

7. Let A be a $n \times n$ tridiagonal symmetric matrix, its diagonal entries are constant given by $a_{ii} = 10$ for $i = 1, \cdots, n$. The superdiagonal entries are random numbers uniformly sampled in the interval $[-5 + 10^{-8}, 5 - 10^{-8}]$. Is it possible that A is singular? If yes, give an example. If not, explain your reason. Can you give a sharp upper bound for the condition number of A? If a backward stable method is used to solve $Ax = b$, where b is an arbitrary non-zero vector on a computer with machine precision 10^{-16}, what is the worst possible error (in the leading magnitude) for the solution? Can you recommend an efficient method to solve the linear system? You must justify your answers.