Topology Comprehensive Exam
Spring 2025

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let ω be an exact 2-form. Show that ω^n is exact for all n. (Here ω^n means the wedge product of ω with itself n times).

2. Let K be the Klein bottle. Recall that K is the surface obtained from $[-1,1]\times[-1,1]$ by identifying $(t,1)$ with $(t,-1)$ and $(1,s)$ with $(-1,-s)$. Give a group presentation for the fundamental group of K. Show that the torus $T^2 = S^1 \times S^1$ is a 2-fold cover of K and compute subgroup of $\pi_1(K)$ corresponding to this cover in terms of the presentation.

3. Let M be a compact manifold without boundary. Show there is no submersion from M to $\mathbf{R}^n$ for any $n > 0$.

4. Let M be a smooth, compact manifold with boundary. Show that there is no smooth map $f : M \to \partial M$ such that $f(x) = x$ for all $x \in \partial M$.

5. Show that the suspension of $\mathbb{R}P^2$ is not a 3-dimensional manifold. Recall that the suspension SX of a space X is the quotient of the product $X \times [0,1]$ obtained by collapsing the subspaces $X \times 0$ and $X \times 1$ to different points.

6. Let M be a 7-dimensional smooth manifold and S be an embedded 1-dimensional submanifold. Show two smooth maps $f, g : S^2 \to (M - S)$ are homotopic in $M - S$ if and only if they are homotopic in M.

7. Let $S = \{(x, y, z, w) \in \mathbf{R}^4 : x^2 + y^2 + z^2 + w^4 = 1\}$ is a manifold and compute its dimension. Consider the vector field on $\mathbf{R}^4$ given by $v(x, y, z, w) = (2y, -2x, 4w^3, -2z)$ (here we are identifying $T_{(x,y,z,w)}\mathbf{R}^4$ with $\mathbf{R}^4$ in the natural way). Show that v restricted to S defines a vector field on S.

8. Let X_k be the quotient space of $\mathbb{R}^2$ by a discrete subset of consisting of $k > 1$ points. Show that X_3 is not homotopy equivalent to a covering space of X_2. Show that X_2 is homotopy equivalent to a covering space of X_3.