Analysis Comprehensive Exam
Spring 2025

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTE:

- All scalars in this exam are real unless explicitly stated otherwise.
- All functions in this exam are (extended) real-valued unless explicitly stated otherwise.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set A is denoted by χ_A.
- The L^1-norm of a measurable function f on a measurable set E is $\|f\|_1 = \int_E |f(x)|\,dx$.

1. Fix $1 \le p,\, q \le \infty$. Let $A = [a_{ij}]_{i,j\in\mathbf{N}}$ be an infinite matrix and for each $i \in \mathbf{N}$ define $a_i = (a_{ij})_{j\in\mathbf{N}}$. Suppose that:

   (a) The series $(Ax)_i = \sum_j a_{ij}x_j$ converges for each vector $x \in \ell^p$ and index $i \in \mathbf{N}$, and

   (b) $Ax = \big((Ax)_i\big)_{i\in\mathbf{N}} \in \ell^q$ for each $x \in \ell^p$.

   Identifying the matrix A with the map $x \mapsto Ax$, prove that A is a bounded mapping from ℓ^p to ℓ^q.

2. Let $\chi_t = \chi_{(t,\infty)}$ be the characteristic function of the interval (t, ∞). Prove that if f and g are measurable functions on $\mathbf{R}$, then $\|f - g\|_1 = \displaystyle\int_{-\infty}^{\infty} \|\chi_t \circ f - \chi_t \circ g\|_1 \, dt$.

3. Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence of measurable functions on $\mathbf{R}$, and let f be a measurable function on $\mathbf{R}$. Suppose that:

   (a) $f_n(x) \to f(x)$ for almost every $x \in \mathbf{R}$,

   (b) $\displaystyle\int_{-\infty}^{\infty} |xf_n(x)|\, dx \le 100$ for every $n \in \mathbf{N}$, and

   (c) $\displaystyle\int_{-\infty}^{\infty} |f_n(x)|^2\, dx \le 100$ for every $n \in \mathbf{N}$.

   Prove that $f_n \in L^1(\mathbf{R})$ for every n, $f \in L^1(\mathbf{R})$, and $\|f - f_n\|_1 \to 0$ as $n \to \infty$.

4. Suppose $A \subset \mathbb{R}$ is a Lebesgue null set, that is, $|A|_e = 0$. Suppose $f : \mathbb{R} \to \mathbb{R}^2$ satisfies $|f(x) - f(y)| \leq \sqrt{|x - y|}$ for $x, y \in \mathbb{R}$. Prove that $f(A)$ is a Lebesgue null set in $\mathbb{R}^2$.

5. (5 points each part) Let (X, A, μ) be a finite measure space. Let E_k be any measurable subsets of X such that $\mu(E_k) \to 0$ as $k \to \infty$. For every $x \in X, N \in \mathbb{N}$, put

$$F_N(x) = \frac{1}{N}|\{n \in \{1, 2, \ldots, N\} : x \in E_n\}|$$

where $|S|$ denotes the cardinality of the set S.

(a) Show that F_N converges to 0 in measure.

(b) Is it necessarily true that $\lim_{N\to\infty} F_N(x) = 0$ μ-almost everywhere? (Either provide a proof or a counterexample.)

6. Suppose $f \in L^2((0,\infty))$. Prove that the function

$$G(t) = \int_0^\infty \frac{f(x)\sin(tx^2)}{1+tx}dx$$

is continuous on $(0,\infty)$.

7. Let E be a measurable subset of $\mathbb{R}$ of finite Lebesgue measure. Let $f(x)$ be any continuous 1-periodic function on the line. Show that $\int_E f(tx)\,dx$ tends to $|E|\int_{[0,1]} f\,dx$ as $t \to \infty$.

8. A measurable function $f : [0,1] \to [1,\infty]$ satisfies $|\{f > y\}| \le y^{-2}$ for all $y > 0$. What is the largest possible value of $\int_{[0,1]} f(x)\,dx$?

   (This means prove an upper bound (7 points) and show by means of an example that it is sharp (3 points).)