Algebra Comprehensive Exam
Spring 2025

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

Notation: Recall that if G is a group and H is a subgroup of G, then $[G : H]$ refers to the index of H in G. In other words, $[G : H]$ is the number of left cosets (or equivalently, right cosets) of H in G.

1. Let G be a finite group and let H be a subgroup of G. If every prime p dividing $|H|$ is at least $[G : H]$, prove that H is a normal subgroup of G.

2. Let p be a prime number. If G is a non-abelian group of order p^3, show that G has exactly $p^2 + p - 1$ conjugacy classes. [**Hint:** Use the fact that the center of G is contained in the centralizer of every element of G.]

3. Show that $\mathbf{Z}[x]$ and $\mathbf{Z}[x, y]/(xy - 1)$ are not isomorphic as rings.

4. Prove that the ring $R = \mathbf{Z}[\sqrt{-11}]$ is not a unique factorization domain by exhibiting two different factorizations of some element into irreducible elements. (Make sure to prove that the factorizations are not equivalent to one another, and that the factors really are irreducible.)

5. Let $L/\mathbf{Q}$ be a Galois extension of order 64. Show that there exists an integer m, which is not a perfect square, such that $\sqrt{m} \in L$.

6. Let L be a subfield of $\mathbf{C}$ such that $[L : \mathbf{Q}]$ is odd and $L/\mathbf{Q}$ is Galois. Show that $L \subset \mathbf{R}$. (In this problem, the notation $[L : \mathbf{Q}]$ refers to the degree of L over $\mathbf{Q}$ as a field extension.)

7. Find the number (up to isomorphism) of $\mathbf{Z}$-modules A with at most three generators such that all $a \in A$ satisfy $6a = 0$.

8. Let A be an $n \times n$ matrix with complex entries and linearly independent eigenvectors $v_1, \ldots, v_n$. Let S be the set of all $n \times n$ complex matrices with the same eigenvectors as A.

    1. Show that S is a vector space of dimension n over $\mathbf{C}$.
    2. Show that S is a commutative ring under the usual matrix addition and multiplication, and that S is isomorphic to $\mathbf{C}^n$ with coordinate-wise addition and multiplication.