Discrete Mathematics Comprehensive Exam
Spring 2025

Student Number: ☐

Instructions: Complete **exactly 5** of the given 6 problems and **circle** their numbers below. The uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

All graphs in the following problems are finite and simple.

1. Let G be a 2-connected plane graph. For each $v \in V(G)$, let
$$\sigma(v) := 1 - d(v)/2 + \sum_{\{f \in F(G): v \in V(f)\}} 1/|V(f)|,$$
where $F(G)$ is the set of facial cycles in G. (1) Show that $\sum_{v \in V(G)} \sigma(v) = 2$. (2) Suppose $v \in V(G)$, $d(v) \geqslant 5$, and $\sigma(v) > 0$. Describe the possible lengths of facial cycles containing v.

2. Let H be a fixed graph. Show that there exists a constant C_H such that if G is any graph not containing H as a minor then $\chi(G) \leqslant C_H$.

3. Let t be a positive integer and let G be the graph obtained from K_{2t} by removing a perfect matching. Show that the list chromatic number of G is at most t.

4. A tournament is an orientation of a complete graph. A tournament is *transitive* if for all triples u, v, w with oriented edges $u \to v$ and $v \to w$, the third edge is oriented $u \to w$. Prove that for all sufficiently large n, there is a tournament on $\lfloor 2^{n/10} \rfloor$ vertices such that none of its subtournaments of size n are transitive.

5. For $n \geqslant 1$, define $C \subseteq \{1,2,3\}^n$ to be *trifference* if for any $x, y, z \in C$, there is some coordinate $i \in [n]$ for which $\{x_i, y_i, z_i\} = \{1,2,3\}$. Prove that for n sufficiently large, there is a trifference set of size at least $(\frac{27}{21})^{\frac{n}{2}}$.

6. If $t \leqslant \frac{n}{8} - \sqrt{n}\log n$, prove that the Erdős-Rényi random graph $G(n, 1/2)$ contains a copy of the complete bipartite graph $K_{3,t}$ with high probability.