

# Topology Comprehensive Exam

## Fall 2025

Student Number:

*Instructions:* Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1      2      3      4      5      6      7      8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. For which  $r \in \mathbf{R}$  is  $\phi^{-1}(r)$  a smooth manifold, and when it is, what is its dimension when  $\phi : \mathbf{R}^3 \rightarrow \mathbf{R} : (x, y, z) \mapsto x^2 + 5y^2 - 3z^2$ . (This means if  $\phi^{-1}(r)$  is a manifold, prove it, and if  $\phi^{-1}(r)$  is not a manifold, prove that too.)
2. Let  $l_x, l_y$ , and  $l_z$  be the coordinate axes in  $\mathbf{R}^3$  and set  $W = \mathbf{R}^3 - (l_x \cup l_y \cup l_z)$ . Compute the fundamental group of  $X$ .
3. Suppose that  $M$  is a non-empty, compact, oriented  $n$ -manifold without boundary. Let  $\omega$  be an  $n$ -form on  $M$  and  $v$  be a vector field on  $M$ . Show that the  $n$ -form  $\mathcal{L}_v\omega$  is zero at some point of  $M$ . Here  $\mathcal{L}_v\omega$  is the Lie derivative of  $\omega$  in the direction of  $v$ .  
Hint: Recall Cartan's formula for the Lie derivative.
4. Let  $M$  and  $N$  be smooth, compact, connected manifolds of the same dimension  $n$ . Suppose  $f : M \rightarrow N$  is a smooth map and there is a regular value  $y \in N$  such that  $f^{-1}(y)$  consists of 3 points. Show that  $f$  is surjective.
5. Let  $X$  be a manifold  $p : \tilde{X} \rightarrow X$  its universal covering space, that is  $\tilde{X}$  is a covering space that is simply connected. If  $\tilde{X}$  is compact, then show that the fundamental group of  $X$  is finite.
6. For any two compact, connected, non-empty smooth manifolds  $M$  and  $N$ , show that any submersion  $f : M \rightarrow N$  is surjective.
7. Let  $Q^n$  be the quotient space of  $\mathbf{R}^n$  by the antipodal involution, that is,  $u, v \in \mathbf{R}^n$  are identified in  $Q$  if and only if  $u = -v$ . Show that  $Q^2$  is homeomorphic to  $\mathbf{R}^2$  and  $Q^3$  is not a topological manifold.
8. Let  $S$  be a smoothly embedded circle in  $\mathbf{R}P^2$ . Suppose that  $X$  is the space obtained by gluing two copies of  $\mathbf{R}P^2$  along the identity map of  $S$ . List all possible (isomorphism classes of) the fundamental group of  $X$ . Explain why every group on your list is the fundamental group of some  $X$  as above, and conversely, the fundamental group of every such  $X$  is on your list.



















