Probability Comprehensive Exam
Fall 2025

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{F}_0 \subset \mathcal{F}$ be an algebra (a.k.a. field) such that $\sigma(\mathcal{F}_0) = \mathcal{F}$; that is, $\mathcal{F}_0$ generates the sigma-algebra (a.k.a. sigma-field) $\mathcal{F}$. Prove that if $A \in \mathcal{F}$ and $\epsilon > 0$, there exists $B \in \mathcal{F}_0$ such that $\mathbb{P}(A \Delta B) < \epsilon$. Here, $A \Delta B$ is the symmetric difference of the sets A and B: $A \Delta B = \{\omega \in \Omega : \mathbf{1}_A(\omega) \neq \mathbf{1}_B(\omega)\}$.

2. Let (X_n) be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{E}X_n = \text{Var } X_n = n$ for all n.

   (i) If (X_n) is monotone nondecreasing (that is, $X_n(\omega) \leq X_{n+1}(\omega)$ for all n and all $\omega \in \Omega$), show that $X_n \to +\infty$ almost surely.

   (ii) Must $X_n \to +\infty$ almost surely if we do not assume that (X_n) is monotone nondecreasing? Prove or give a counterexample.

3. Let (X_n) be an i.i.d. sequence of random variables with common probability density function f, given by
$$f(x) = \begin{cases} \frac{1}{2x} & \text{if } x \in [e^{-1}, e] \\ 0 & \text{otherwise.} \end{cases}$$
For $n \geq 1$, define $M_n = \prod_{k=1}^n X_k$. Show that $M_n^{1/\sqrt{n}}$ converges in distribution as $n \to \infty$, and identify the limit.

4. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_n)_{n \geq 1}$ be a sequence of centered Gaussian/Normal random variables each with variance σ_n^2 such that $\lim_{n \to +\infty} \sigma_n^2 = \sigma^2$, $0 < \sigma^2 < +\infty$, and let $(Y_n)_{n \geq 1}$ be a sequence of random variables each having law $\mu_{Y_n} = \sum_{k=0}^{+\infty} q_k^{(n)} \delta_k$, with $\lim_{n \to +\infty} q_1^{(n)} = 1$. Show that the sequence $(X_n Y_n)_{n \geq 1}$ converges in law towards a random variable that you will identify.

5. If X is a random variable, define $M(X) = \{r \geq 0 : \mathbb{E}|X|^r < \infty\}$. Let $\alpha > 0$.

   (i) Find an example of a random variable X such that $M(X) = [0, \alpha)$.

   (ii) Find an example of a random variable X such that $M(X) = [0, \alpha]$.

6. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let X be a random variable with finite second moment, and let $\text{Var}X$ denote its variance.

   (i) Show that for any $x > 0$,
$$\mathbb{P}\left(X - \mathbb{E}X \geq x\right) \leq \frac{\text{Var}X}{x^2 + \text{Var}X}.$$

   (ii) Let now $(X_n)_{n \geq 1}$ be a sequence of independent random variables with finite second moment and such that $\text{Var}X_n \leq 1$. Let $N = \min\{n \geq 1 : X_n \leq 1\}$. Show that $e^{\lambda N}$ has finite expectation, for $\lambda \in [0, \ln 2)$.

7. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_n)_{n\geq 1}$ be a sequence of non-negative independent and identically distributed (iid) random variables such that $\mathbb{P}(X_1 = 0) < 1$.

   (i) What is the probability that the series $\sum_{n=1}^{+\infty} X_n$ converges?

   (ii) Let $\mathbb{E}(X_1) = +\infty$. Show that $\mathbb{P}$-almost surely, there exists a subsequence of $(X_n/n)_{n\geq 1}$ diverging to infinity.

8. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed (iid) Bernoulli random variables with parameter $1/2$ and let $Y_n = X_n X_{n+1}$, $n \geq 1$.

   (i) Show that there exists a non-negative integer m such that for any $n \geq 1$ and $k \geq m+1$, the random variable Y_{n+k} is independent of $Y_1, \ldots, Y_n$.

   (ii) Let $S_n = \sum_{k=1}^{n} X_k X_{k+1}$. Show that $(S_n/n)_{n\geq 1}$ converges in probability towards a limit that you'll indentify.